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THE MARKOV INEQUALITY FOR SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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The purpose of this paper is to prove the following theorem.

THEOREM. *Let $S_n = X_1 + \cdots + X_n$ be a sum of n independent, non-negative random variables with means*

$$\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n) = (EX_1, \cdots, EX_n)$$

$$N = \nu_1 + \cdots + \nu_n;$$

and, for each $\lambda > N$, let

$$\psi_n(\lambda; \boldsymbol{\nu}) = \sup P(S_n \geqq \lambda),$$

where the supremum is taken over all such S_n. (We ignore $\lambda \leqq N$ since the supremum is trivially one.) *Then,*

$$\lambda \geqq [\max(4, n-1)]N \Rightarrow \psi_n(\lambda; \boldsymbol{\nu}) = 1 - \prod_{1 \leqq i \leqq n} (1 - \nu_i/\lambda),$$

which is attained if and only if, for each i,

$$P(X_i = \lambda) = \nu_i/\lambda = 1 - P(X_i = 0).$$

Since these X_i's are identically distributed when the means are equal, we have an immediate

COROLLARY. *Let $\{X_i : 1 \leqq i \leqq n\}$ be* i.i.d., *non-negative, with common mean ν. If $\lambda > [\max(4n, (n-1)n)]\nu$, then*

$$P(X_1 + \cdots + X_n \geqq \lambda) \leqq 1 - (1 - \nu/\lambda)^n.$$

Equality holds if and only if $X_i \,\varepsilon\, \{0, \lambda\}$.

We shall present an outline of the proof as a series of lemmas. The first three lemmas show that, to prove the theorem, it suffices to prove the proposition following Lemma 3. The remaining three lemmas constitute a proof of that proposition. After stating the six lemmas, we sketch their proofs. Finally, there is a brief discussion of how the theorem may be improved.

LEMMA 1. *Without loss of generality we may assume that each X_i has at most two mass points—call them a_i and b_i—satisfying:*

$$0 \leqq a_i < \nu_i \leqq b_i \leqq \lambda, \tag{1}$$

$$P(X_i = b_i) = (\nu_i - a_i)/(b_i - a_i) = 1 - P(X_i = a_i).$$

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