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# ON THE MATRIX RENEWAL FUNCTION FOR MARKOV RENEWAL PROCESSES[1]

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**1. Introduction.** A Markov renewal process [7, 8], $[N_1(t), N_2(t), \cdots, N_R(t)]$ is an extension of an ordinary renewal process having considerable practical importance. Whereas the ordinary renewal process describes the number of recurrences (renewals) in the interval $(0, t]$ for a single recurrent class of epochs $\mathcal{E}_1$ of interest, the Markov renewal process describes the recurrence statistics for intermingling classes of epochs $\{\mathcal{E}_j; j = 1, 2, \cdots R\}$ of an underlying semi-Markov process. The process is characterized by a stochastic transition matrix $p_{ij}$ for the Markov chain governing the sequence of successive epochs, and a matrix of probability distributions $F_{ij}(x)$ for the time elapsing between epochs of class $\mathcal{E}_i$ and epochs of class $\mathcal{E}_j$, whenever an $\mathcal{E}_j$ epoch follows an $\mathcal{E}_i$ epoch. We adopt the convention that an epoch of a given class may be succeeded by another epoch of that class.

Let $N_{ij}(t)$ be the number of epochs of class $\mathcal{E}_j$ appearing in the interval $(0, t]$, when it is known that at $t = 0$ there was an epoch of class $\mathcal{E}_i$. Let $H_{ij}(t) = E[N_{ij}(t)]$ and let $\mathbf{H}(t)$ be the $R \times R$ matrix with elements $H_{ij}(t)$. We will call this matrix the matrix renewal function.

When $R = 1$, $N_{11}(t)$ is the ordinary renewal process, and $H_{11}(t) = H(t)$ is the renewal function. It is well known [6] that if an interval distribution $F(x)$ has finite first and second moments $\mu_1$ and $\mu_2$, and does not have arithmetic support, then the associated renewal function $H(t)$ has the behavior

$$H(t) = \mu_1^{-1}t + \tfrac{1}{2}\mu_1^{-2}(\mu_2 - 2\mu_1^2) + \epsilon(t) \tag{1.1}$$

where $\epsilon(t)$ is bounded and goes to zero as $t \to \infty$.

The literature on semi-Markov processes has dealt largely with the theoretical structure of such processes, and pathology distinguishing such processes from Markov chains, in continuous time. The statistics for such processes, e.g. the mean and variance of the renewal time between epochs of the same class, and the mean passage time from an epoch of class $\mathcal{E}_i$ to an epoch of class $\mathcal{E}_j$, have received less attention.[2] For finite semi-Markov processes ($R < \infty$), much of this statistical information is available from the following direct analogue of equation (1.1).

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[2] Other procedures and results may be found in W. S. Jewell, Markov–renewal programming I, II, *Operations Res.*, **11** (1963) 938–971; Kshirsagar and Gupta, Asymptotic values of the two moments in Markov renewal processes, *Biometrika*, **54** (1967); and P. J. Schweitzer, Perturbation theory and Markovian decision processes, MIT ScD Thesis, Department of Physics, 1965. I am indebted to D. R. Cox and P. J. Schweitzer for these references.