The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 321–325

ON A CHARACTERIZATION OF THE WIENER PROCESS BY CONSTANT REGRESSION

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1. Introduction. Recently Cacoullos [1] has proved the following theorem which characterizes Normal Distribution by constant regression of one linear statistic on another linear statistic.

THEOREM 1.1. *Let X_i, $1 \leqq i \leqq n$ be a random sample from a univariate population with non-degenerate distribution function $F(x)$, and assume that $F(x)$ has moments of every order. Consider the linear statistics*

$$U = a_1 X_1 + \cdots + a_n X_n; \qquad V = b_1 X_1 + \cdots + b_n X_n$$

where a_i, $1 \leqq i \leqq n$; b_i, $1 \leqq i \leqq n$ have the property that $\sum_{i=1}^n a_i b_i = 0$ implies that $\sum_{i=1}^n a_i b_i^k \neq 0$ for all $k > 1$. Then U has constant regression on V, i.e.,

$$E[U \mid V] = E[U] \quad \text{a.e.}$$

if and only if (1) *the population distribution F is normal, and* (2) $\sum_{i=1}^n a_i b_i = 0$.

In this paper we derive a similar result for a characterization of the Wiener process. We would like to mention that a theorem similar to Theorem 1.1 has also been obtained by Rao [4].

2. Preliminaries. Let $T = [A, B]$. We shall consider stochastic processes $\{X(t), t \in T\}$ which have finite moments of all orders. In particular, $\{X(t), t \in T\}$ will be a stochastic process of the second order. Let $a(\cdot)$ be a function which is continuous on $[A, B]$, and suppose that the mean function $m(t) = E[X(t)]$ and the covariance function $r(s, t) = E[X(t)X(s)] - E[X(t)]E[X(s)]$ are of bounded variation in $[A, B]$. It can be shown that the integral

$$\int_A^B a(t)\,dX(t) \tag{2.1}$$

exists as the limit in the mean (lim) of the corresponding Riemann-Stieltjes sums.

A stochastic process $\{X(t), t \in T\}$ is said to be a homogeneous process with independent increments if the distribution of the increments $X(t+h) - X(t)$ depends only on h but is independent of t, and if the increments over non-overlapping intervals are stochastically independent. The process is said to be continuous if $X(t)$ converges in probability to $X(s)$ as t tends to s for every $s \in T$. Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments. Let $\varphi(u; h)$ denote the characteristic function of $X(t+h) - X(t)$. It is well known that $\varphi(u; h)$ is infinitely divisible and $\varphi(u; h) = [\varphi(u; 1)]^h$. See Lukacs [2].

A homogeneous process $\{X(t), t \in T\}$ with independent increments is called a Wiener process if the increments $X(t+h) - X(t)$ are normally distributed with variance proportional to h.

Received April 2, 1969; revised July 28, 1969.