The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 305–309

NOTES

A BOUND FOR THE DISTRIBUTION OF THE MAXIMUM OF CONTINUOUS GAUSSIAN PROCESSES

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Let $X(t)$ be a real valued separable Gaussian process on $[0, 1]$. $E\{X(t)^2\}^{\frac{1}{2}} \leqq \Gamma$ and $E\{(X(t)-X(s))^2\} \leqq \psi(|t-s|)$, where ψ is assumed to be continuous and non-decreasing on $[0, 1]$. Consider the random function $\|X\|_\infty = \sup_{t\in[0,1]}|X(t)|$. We shall give an upper bound for the "tail" of the probability distribution of $\|X\|_\infty$.

The major result in this paper is a lemma that is very close to Fernique's lemma [1]. In fact the motivation for this work was to find a proof of this important lemma. As far as the author knows, none has been published or is otherwise available. Fernique's use of the lemma is to provide sufficient conditions for the continuity of Gaussian processes. A proof of his continuity result is given by Dudley [2]; our proof of the lemma provides an alternate proof of this result. However, the lemma has other significant applications, two of which will be mentioned below. The proof in this paper was suggested by Nisio's proof of her Theorem 1, [3]. From the corollary to the lemma a simple proof of Nisio's result will be obtained.

Our lemma is presented differently from Fernique's lemma because in many cases it yields sharper results. In the study of Hölder conditions for Gaussian processes the lemma presented in this paper enables us to improve previous results of the author [4] obtained originally by using Fernique's form of the lemma.

The following two conditions on the processes will be used:

(A) $\int_1^\infty \psi(e^{-x^2})\,dx < \infty$;

(B) $\psi^2(h)\log 1/h$ decreases monotonically as h decreases to zero from the right.

All processes studied will be assumed to satisfy condition (A) whereas results will be given for the cases when condition (B) applies and when it does not apply. In [4] it is pointed out that condition (B) is widely satisfied when (A) is satisfied. The expression in (B) occurs in the study of uniform Hölder conditions for Gaussian processes.

We now prove the lemma.

LEMMA. *Let $X(t)$ be real valued, separable Gaussian process on* $[0, 1]$. *Define ψ as above, and assume condition* (A) *is satisfied. Let $c(p)$ denote n^{2^p} for n* a fixed integer, $n > 1$; *let $a \geqq (2\beta\log n)^{\frac{1}{2}}$ where $\beta \geqq 2$. Then*

$$(1)\qquad P\{\|X\|_\infty \geqq a\Gamma + b(2\beta)^{\frac{1}{2}}\textstyle\sum_{p=1}^\infty \psi(c(p)^{-1})(\log c(p))^{\frac{1}{2}}\}$$
$$\leqq n^2\int_a^\infty e^{-x^2/2}\,dx + G(\beta, n),$$

Received September 30, 1968.