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ON THE WAITING TIME IN THE QUEUING SYSTEM GI/G/1

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1. Introduction. This paper deals with the waiting time $X(t)$ in the queuing model $GI/G/1$. $X(t)$ is defined as the time needed to complete the serving of all those units which are present in the system at time t. In order to obtain information about the distribution of $X(t)$, we use the auxiliary variable $Y(t)$, defined as the time between t and the first arrival after t. The vector $(X(t), Y(t))$ forms a Markov process. We consider the distribution function $L_{y_0}^{x_0}(t; y, x) = P_r\{Y(t) \leqq y, X(t) \leqq x \mid Y(0) = y_0, X(0) = x_0\}$ and obtain in the case $x_0 = 0$, $y_0 \geqq 0$ a closed expression for

$$\hat{L}_{y_0}^{x_0}(\theta; s, w) = \int_{t=0}^{\infty} \int_{y=0}^{\infty} \int_{x=0-}^{\infty} e^{-\theta t} e^{-sy} e^{-wx} L_{y_0}^{x_0}(t; dy, dx)\, dt.$$

This contains as a special case an expression for $\hat{L}_{y_0}^{x_0}(\theta; 0, w)$, which is the Laplace-Stieltjes transform with respect to x and the Laplace transform with respect to t of the distribution function of $X(t)$, thus determining this distribution function completely. The results are valid for arbitrary service-time distribution function $B(t)$ concentrated on $[0, \infty)$ and for any interarrival-time distribution function $A(t)$ with $A(0) = 0$. Our analysis is based on the method of stages, described in [4]. This method exploits the fact that every distribution function $F(t)$ concentrated on $[0, \infty)$ can be approximated weakly as u tends to infinity by distribution functions

$$F_u(t) \doteq F(0) + \sum_{k=1}^{\infty} \{F(k/u) - F((k-1)/u)\} E_u^{k*}(t),$$

where $E_u^{k*}(t)$ is the k-fold convolution of the distribution function $1 - e^{-ut}$ (see [4]).

The results seem to be new. Keilson and Kooharian ([3]) investigated the system and derived expressions for Laplace transforms of the regeneration and server occupation time distributions. They were led to Wiener-Hopf type equations, and this aspect of the problem appears in our analysis as well, although we do not use the corresponding techniques. Takács ([5]) has derived the limiting distribution of $X(t)$ as t tends to infinity, using the same auxiliary random variable $Y(t)$ that we use.

The method of stages, as applied in this paper, consists essentially of associating to the Markov process $(X(t), Y(t))$ a family of discrete-state Markov processes, whose members approximate it. The theory of this procedure is intended to be presented in a wider context at a future time and we feel therefore justified in omitting the proof for its validity in this paper. Unfortunately, our method does not cover the case $x_0 > 0$, corresponding to a non-empty queue at $t = 0$. Although we are able to derive the desired expressions in this case for the approximating system, their unwieldy appearance makes the limiting procedure seem intractable.

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