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# AN EXTENSION OF WILKS' TEST FOR THE EQUALITY OF MEANS

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**1. Introduction.** When $x = (x_1, \cdots, x_p)$ has a multivariate normal distribution with mean vector $\theta$ and covariance matrix $\Sigma$, then Hotelling's $T^2$ test may be used for testing $\theta = 0$ versus $\theta \neq 0$. However, when part of the mean vector is known, i.e., $\dot{\theta} = (\theta_1, \cdots, \theta_k) = 0$, and we wish to test that the remaining part is zero, i.e., $\ddot{\theta} = (\theta_{k+1}, \cdots, \theta_p) = 0$ against $\ddot{\theta} \neq 0$, then the $T^2$-statistic should not be used. The likelihood ratio test is given by $(1+T_k^2)/(1+T_p^2)$, where $T_m^2$ is the usual $T^2$-statistic in which the first $m$ variates is used. This test, its properties, and alternative tests have been studied by a number of authors (all the references may be obtained from Olkin and Shrikhande (1954) and Kabe (1965)).

When the variables are interchangeable with respect to variances and covariances—the intraclass correlation model—the test of the hypothesis $\theta_1 = \cdots = \theta_p$ versus $-\infty < \theta_j < \infty$, $j = 1, \cdots, p$, was obtained by Wilks (1946). If we know that $\theta_1 = \cdots = \theta_k$, and wish to test for the equality of all the means, the test statistic should be a variant of that obtained by Wilks. The present paper deals with this problem.

Define $\Sigma_I = \sigma^2[(1-\rho)I + \rho e'e]$, where $e = (1, \cdots, 1)$, and regions

$$\begin{aligned} &\omega_1 = \{\theta, \Sigma_I : \theta_1 = \cdots = \theta_p, \Sigma_I > 0\}, \\ (1.1)\quad &\omega_2 = \{\theta, \Sigma_I : \theta_1 = \cdots = \theta_k, \theta_{k+1} = \cdots = \theta_p, \Sigma_I > 0\}, \\ &\omega_3 = \{\theta, \Sigma_I : \theta_1 = \cdots = \theta_k, -\infty < \theta_j < \infty, j = k+1, \cdots, p, \Sigma_I > 0\}, \\ &\omega_4 = \{\theta, \Sigma_I : -\infty < \theta_j < \infty, j = 1, \cdots, p, \Sigma_I > 0\}, \end{aligned}$$

where $\Sigma_I > 0$ means that $\Sigma_I$ is positive definite. The hypothesis $(\theta, \Sigma_I) \in \omega_1$ versus $\omega_4$ is the Wilks hypothesis; we now consider testing $\omega_1$ versus $\omega_2$, $\omega_1$ versus $\omega_3$, and $\omega_2$ versus $\omega_3$. For each problem the likelihood ratio statistic (LRS) and its non-central distribution are obtained. In order to avoid degeneracies, we assume $p-1 \geqq k \geqq 2$. However, note that the test of $\omega_3$ versus $\omega_4$, for example, reduces to Wilks hypothesis when $p = k$.

**2. Derivation of tests.** Given a sample of size $N$, we have the sufficient statistic $(\bar{x}, S)$, where $\bar{x}$ is the sample mean and $S$ is the matrix of sample cross-products. Then $\bar{x}$ and $S$ are independently distributed, $\mathscr{L}(\bar{x}) = N(\theta, \Sigma/N)$, $\mathscr{L}(S) = W(\Sigma, p, n)$, where $n = N-1$, i.e., $\bar{x}$ and $S$ have the joint density function

$$p(\bar{x}, S) = c\,|\Sigma|^{-N/2}\,|S|^{(n-p-1)/2} \exp\{-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}[S + N(\bar{x}-\theta)'(\bar{x}-\theta)]\},$$

where $c = N^{p/2}[2^{pN/2}\pi^{p(p+1)/4}\prod_1^p \Gamma(\frac{1}{2}(n-i+1))]^{-1}$.

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