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JOINT DISTRIBUTION OF THE EXTREME ROOTS OF A COVARIANCE MATRIX[1]

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1. Introduction and result. The purpose of this note is to find the joint distribution and the distribution of the ratio of the largest root and the smallest root of a sample covariance matrix when the population covariance matrix is a scalar matrix, $\mathbf{\Sigma} = \sigma^2\mathbf{I}$. The main result in this paper is the following

THEOREM. *Let* $\mathbf{S}$ *be a* $(p \times p)$ *matrix having a Wishart distribution* $\mathbf{W}(p, n, \mathbf{I})$, *and* $\lambda_1, \lambda_2, \cdots, \lambda_p$ $(\infty > \lambda_1 > \lambda_2 > \cdots > \lambda_p > 0)$ *be the latent roots of the matrix* $\mathbf{S}$. *Then the distribution of* $x = 1-\lambda_p/\lambda_1$ *is given by*

$$\begin{aligned} f(x) = C(p)\cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}(\Gamma(pn/2+k)/p^k k!) \\ \cdot \textstyle\sum_{s=0}^{\infty}(((p-1)(p+2)/2+k+s)/s!)x^{(p-1)(p+2)/2+k+s-1} \\ \cdot \textstyle\sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta}(((p+1-n)/2)_{\sigma}((p+2)/2)_{\delta}/(p+1)_{\delta})C_{\delta}(\mathbf{I}_{p-1}) \end{aligned} \tag{1}$$

where $1 > x > 0$, the subscript κ is usual partition of the integer k not more than p parts, the subscript σ and δ are the partitions of the integers s and $k+s$ into not more than $p-1$ parts respectively, the summation $\sum_{\sigma,\delta}$ is over all combinations of these partitions, and the constant

$$C(p) = \pi^{p/2}B_{(p-1)}(p/2, (p+2)/2)/P^{pn/2}\Gamma(p/2)\Gamma_p(n/2).$$

We notice that g-coefficients come from

$$C_{\kappa}(\mathbf{L})C_{\sigma}(\mathbf{L}) = \textstyle\sum_{\delta} g_{\kappa,\sigma}^{\delta}\, C_{\delta}(\mathbf{L})$$

tabulated up to the 7th degree in Khatri and Pillai [2].

Consider the sphericity test, $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$, where σ^2 is unspecified. For the test criteria, we may suggest the likelihood ratio criterion of the geometric mean and the arithmetic mean, $\prod\lambda_i^{1/p}/(\sum\lambda_i/P)$, and also the ratio, $\lambda_p/\lambda_1 \uparrow = 1-X$, of the largest root λ_1 and the smallest root λ_p, identically $(\lambda_1\lambda_p)^{\frac{1}{2}}/((\lambda_1+\lambda_p)/2)$. The joint distribution of the roots λ_1 and λ_p given by (8) is associated with the problems of finding confidence bounds. (See Roy and Gnanadesican [3], and Anderson [1].)

2. Joint distribution and the distribution of the ratio of the largest root and the smallest root. Let $\mathbf{S}$ be the same matrix as before. The joint distribution of the latent roots $\lambda_1, \cdots, \lambda_p$ of the matrix $\mathbf{S}$ is written as follows

$$f_1(\lambda_1, \cdots, \lambda_p) = C\left|\mathbf{\Lambda}\right|^{(n-p-1)/2}\exp(\operatorname{tr}(-\tfrac{1}{2}\mathbf{\Lambda}))\textstyle\prod_{i<j}(\lambda_i-\lambda_j) \tag{2}$$

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