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ON THE WEAK CONVERGENCE OF PROBABILITY MEASURES[1]

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1. Introduction. Let $\mathscr{X}$ be a set carrying a σ-field $\mathscr{A}$ and a family of probability measures $\{P_\theta : \theta \in \Theta\}$. Let L be the smallest L-space which contains all the P_θ, $\theta \in \Theta$. This is the smallest linear space which is complete for the usual total variation norm and contains all the measures smaller than linear combinations of the P_θ. It is a Banach space with a dual M which contains, often properly, the space of equivalence classes of bounded $\mathscr{A}$-measurable functions.

Upon replication of the experiment $\{\mathscr{X}, \mathscr{A}, P_\theta ; \theta \in \Theta\}$ the relevant measures are the product measures $P_\theta \otimes P_\theta$ on the product space $\{\mathscr{X} \times \mathscr{X}, \mathscr{A} \times \mathscr{A}\}$. Several problems about the existence of "consistent tests", "sequential discrimination" and similar subjects lead to the following question:

If μ is a measure in the $w(L, M)$ closure $\bar{S}$ of the set $S = \{P_\theta ; \theta \in \Theta\}$ is the product measure $\mu \otimes \mu$ in the closure of $\{P_\theta \otimes P_\theta ; \theta \in \Theta\}$?

It is an easy consequence of a theorem of Dunford and Pettis (see [1], [2]) that the answer to this question is "yes" if the set $\{P_\theta ; \theta \in \Theta\}$ is $w(L, M)$ relatively compact in L. In particular if there is a *sequence* $\{P_{\theta_n}\}$ which converges to μ then $P_{\theta_n} \otimes P_{\theta_n}$ converges to $\mu \otimes \mu$.

The answer is also "yes" if $\mathscr{X}$ is a countable set. Finally, the answer is "yes" if the set $S = \{P_\theta ; \theta \in \Theta\}$ is convex, since in this case the strong closure of S coincides with its $w(L, M)$ closure. The purpose of the present note is to show that there do exist families $\{P_\theta ; \theta \in \Theta\}$ for which the answer is "no". In fact we shall demonstrate the existence of a *countable* collection $S = \{P_\theta ; \theta \in \Theta\}$ and a measure μ such that $\mu \in \bar{S}$ but such that $\mu \otimes \mu$ is remote from the closure of the convex hull of the set $\{P_\theta \otimes P_\theta ; \theta \in \Theta\}$.

In the usual statistical context one considers not only the products $P_\theta \otimes P_\theta$ but also for each integer n the experiment $\mathscr{E}_n$ consisting of taking n independent identically distributed observations whose distribution is either μ or one of the $P_\theta ; \theta \in \Theta$. The available σ-field for $\mathscr{E}_n$ is the product $\mathscr{A}^n$ of n copies of $\mathscr{A}$. The measures are the corresponding products μ^n or $P_\theta{}^n$. Let then $\mathscr{P}$ be the set of all probability measures on $\{\mathscr{X}, \mathscr{A}\}$. For each n one can use $\mathscr{A}^n$, or equivalently the space $\mathscr{V}_n$ of equivalence classes of bounded $\mathscr{A}^n$-measurable functions to define a uniform structure $\mathscr{U}_n$ on $\mathscr{P}$. Let also $\mathscr{U}$ be the uniform structure defined by $\bigcup_n \mathscr{V}_n$. Let $\hat{S}$ be the closure of $S = \{P_\theta ; \theta \in \Theta\}$ in $\mathscr{P}$ for the structure $\mathscr{U}$. Let us say that there exist uniformly consistent tests of μ against S if there exist functions $\varphi_n \in \mathscr{V}_n$, $0 \leqq \varphi_n \leqq 1$ such that $E[\varphi_n \mid \mu] \to 1$ while $\sup_\theta \{E[\varphi_n \mid P_\theta]; \theta \in \Theta\} \to 0$.

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