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A CORRESPONDENCE BETWEEN BAYESIAN ESTIMATION ON STOCHASTIC PROCESSES AND SMOOTHING BY SPLINES

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1. Introduction. Let

$$L = \sum_{j=0}^{m} a_j D^j \qquad m > 0, a_m \neq 0 \tag{1.1}$$

be a linear differential operator with real constant coefficients and denote the adjoint operator by

$$L^* = \sum_{j=0}^{m} a_j(-D)^j. \tag{1.2}$$

Let $\{t_i : i = 1, 2, \cdots, n\}$ be a set of distinct real constants.

DEFINITION. An L-spline with knots $\{t_i\}$ is a function $x \in C^{2m-2}$ for which

$$L^*Lx = 0 \tag{1.3}$$

on each open interval $(-\infty, t_1)$, (t_i, t_{i+1}), (t_n, ∞). Hence an L-spline consists of piecewise solutions of a $2m$th order linear homogeneous differential equation joined at the knots in such a manner as to maintain continuity of all derivatives up to and including the $(2m-2)$th. If we were to require that $x \in C^{2m-1}$, then x would satisfy (1.3) everywhere. Thus, an L-spline can be looked upon as the "most differentiable" function which satisfies (1.3) on the appropriate open intervals without satisfying it everywhere. Although operators of the form (1.1) are sufficiently general for our present purposes, it should be pointed out that L-splines are often defined and studied for other linear differential operators, in which case the domain of definition of x is taken to be a finite closed interval. References [1], [3] and [6] contain extensive bibliographies on splines.

Two common problems for which L-splines are solutions are the following:

(i) *Curve fitting*. Given data $\{(t_j, y_j): j = 1, 2, \cdots, n\}$ to find a function $\hat{x}(t)$ which minimizes

$$\int_{-\infty}^{\infty} (Lx)^2 \, dt \tag{1.4}$$

among all functions x in a certain class for which

$$x(t_j) = y_j \qquad j = 1, 2, \cdots, n. \tag{1.5}$$

If (1.4) is interpreted as a measure of the roughness of x, then $\hat{x}$, if it exists, is the smoothest interpolator to the data.

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