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A NOTE ON THE EXISTENCE OF THE WEAK CAPACITY FOR CHANNELS WITH ARBITRARILY VARYING CHANNEL PROBABILITY FUNCTIONS AND ITS RELATION TO SHANNON'S ZERO ERROR CAPACITY[1]

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0. Introduction. In [7] Kiefer and Wolfowitz stated the coding problem for channels with arbitrarily varying channel probability functions (a.v.ch.)—including cases with side information—for pure codes and maximal errors, and gave necessary and sufficient conditions for the channel rate to be positive. (A detailed discussion about the different coding problems which arise in the theory of a.v.ch. by using different code concepts and different error concepts is given in [2].)

It was undecided for a long time whether the coding theorem and the weak converse or also the strong converse of the coding theorem hold. (Compare [5] page 566.) If the coding theorem and weak converse hold for a channel then we say that this channel has a *weak* capacity, and if also the strong converse holds then we say that this channel has a *strong* capacity.

In Section 1 we prove that the a.v.ch. considered in [7] have a weak capacity. In case of output alphabets of length 2 an *explicit* formula for the strong capacity is even known [3]. One would like to have this sharper result for general finite output alphabets, but, while awaiting a solution of this difficult problem, it would be of interest to know that at least the weak capacity exists. The disadvantage of our method is obviously that it does not lead to a formula for the capacity. However, already for stationary semicontinuous compound channels a reasonable formula for the weak capacity is unknown [6]. Moreover, in Section 2 we shall show that Shannon's zero-error-capacity problem for the discrete memoryless channel (d.m.c.) is equivalent to finding an explicit formula for the capacity of a special a.v.ch. defined by a set of stochastic 0–1 matrices. Therefore an explicit formula for the weak capacity of an a.v.ch. would imply the solution of Shannon's problem, which is known to be of a graph theoretical nature and very difficult (cp. [8], [4]). The close relation between the two problems might give some hope of finding explicit formulas for the error-capacity of an a.v.ch. also in other cases of an alphabet length for which the zero-error capacity is known.

Our method for proving the existence of the weak capacity for an a.v.ch. applies to infinite alphabets and other channels than a.v.ch. (Corollary 1 and Corollary 2 in Section 1).

1. The existence of the weak capacity for a.v.ch. Let $X^t = \{1, \cdots, a\}$, $Y^t = \{1, \cdots, b\}$ for $t = 1, 2, \cdots$ and let $\mathscr{L} = \{w(\cdot\,|\,\cdot\,|s)\,|\,s \in S\}$ be a set of stochastic

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