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THE LIMIT POINTS OF A NORMALIZED RANDOM WALK[1]

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1. Introduction and statement of results. This paper deals with the set of accumulation points of $n^{-\alpha}S_n$ for a one-dimensional random walk S_n, $n \geqq 1$. S_n is called a random walk if $S_n = \sum_{i=1}^n X_i$ for a sequence $\{X_i\}_{i \geqq 1}$ of independent, identically distributed random variables. The (random) set of accumulation points of $n^{-\alpha}S_n$ will be denoted by

$$(1.1) \quad A(S_n, \alpha) = \text{set of accumulation points of } n^{-\alpha}S_n, n \geqq 1 = \bigcap_m \overline{\{n^{-\alpha}S_n : n \geqq m\}}.$$

The bar in the last member of (1.1) denotes closure in the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ with its usual topology. This meaning for a bar over a subset of $\overline{\mathbf{R}}$ will be maintained throughout; F will always denote the common distribution function of the X_i.

The motivation for this study lies in two recent results. Firstly, a condition of K. G. Binmore and M. Katz (private communication) for a point b to be an accumulation point of S_n/n. Secondly, a necessary and sufficient condition of Stone [16] for $+\infty$ (or $-\infty$) to belong to $A(S_n, \frac{1}{2})$.

In Section 2 we first prove that $A(S_n, \alpha)$ is w.p.1[2] equal to a fixed (non-random) closed set $B(\alpha)$. Of course $B(\alpha)$ depends on F, and in fact can be viewed as a characteristic of F. In particular $B(1)$ is a sort of generalized mean; it consists only of the number $\int x\, dF(x)$ whenever this integral is meaningful. Next derive two forms of a necessary and sufficient condition for a point b to belong to $B(\alpha)$ (Theorems 2 and 3). The first form of the conditions and its proof (Theorem 2 with Corollaries 1 and 2) are essentially due to K. G. Binmore and M. Katz. In Section 3 we use these conditions to derive the possible forms of $B(\alpha)$ for $0 < \alpha < \frac{1}{2}$, and in part for $\alpha = \frac{1}{2}$. Specifically we prove

THEOREM 4. *Assume* $F(0) - F(0-) < 1$. *Let* $0 < \alpha < \frac{1}{2}$. *If* $n^{-\alpha}S_n$ *has* w.p.1 *a finite limit point, then* w.p.1 *all real numbers are limit points of* $n^{-\alpha}S_n$ (*i.e., if* $B(\alpha) \cap \mathbf{R} \neq \varnothing$ *then* w.p.1 $A(S_n, \alpha) = \overline{\mathbf{R}}$).

If $\alpha = \frac{1}{2}$ *and* $n^{-\frac{1}{2}}S_n$ *has* w.p.1 *a finite limit point, then* w.p.1 $n^{-\frac{1}{2}}S_n$ *has at least a half line* $[b, \infty]$ *or* $[-\infty, b]$ *as limit points.*

We conjecture that even for $\alpha = \frac{1}{2}$ $B(\frac{1}{2}) = \overline{\mathbf{R}}$ as soon as $B(\frac{1}{2}) \cap \mathbf{R} \neq \varnothing$. If correct this result would be an extension of part of Stone's result in [16]. Indeed, the result of [16] implies that if $B(\frac{1}{2}) \cap \mathbf{R} \neq \varnothing$ then $B(\frac{1}{2})$ contains $+\infty$ and $-\infty$. In Section 4 we sharpen Stone's result in another direction. We prove that if $EX_1^+ = +\infty$ and if $n^{-1}S_n \geqq a$ i.o.[2] w.p.1 for some fixed $a \in \mathbf{R}$, then $\limsup_{n\to\infty} n^{-1}S_n = +\infty$ w.p.1

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[2] w.p.1 = with probability 1; we shall occasionally leave out the expression w.p.1 when there is no risk of confusion. i.o. = infinitely often.