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A CENTRAL LIMIT THEOREM WITH NONPARAMETRIC APPLICATIONS[1]

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Many nonparametric test and confidence interval procedures can be based on statistics of the type

$$S = \sum_{i \in I} \sum_{j \in J} v_{ij},$$

where $v_{ij} = 1$ or 0 depending on whether some other variable u_{ij} is $<$ or $> u_0$, u_0 being an appropriate constant, and where two variables u_{ij} and u_{gh} are independent if no subscript in the (i, j)-pair matches a subscript in the (g, h)-pair. If the number of elements in I and the number of elements in J depend linearly on some index N, we are interested in a simple central limit theorem for statistics S as N increases indefinitely.

Let $\mu = ES$ and $\sigma^2 = \operatorname{var} S$. We then have the following

THEOREM. *A sufficient condition for the asymptotic normality of* $(S-\mu)/\sigma$ *is that* Var S *is of order* N^3. (*We shall express this condition by writing* $\sigma^2 = \Omega(N^3)$.)

PROOF. Let $w_{ij} = v_{ij} - p_{ij}$, where $p_{ij} = P(v_{ij} = 1) = P(u_{ij} < u_0)$, and set

$$W = \sum_{i \in I} \sum_{j \in J} w_{ij}.$$

If μ_r denotes the rth moment of W, $r = 3, 4, \cdots$, we shall show that for $k = 2, 3, \cdots$, $\lim_{N \to \infty} \mu_{2k-1}/\sigma^{2k-1} = 0$ and $\lim_{N \to \infty} \mu_{2k}/\sigma^{2k} = (2k-1)(2k-3) \cdots 3$. These are the moments of the standard normal distribution, so that the theorem follows.

We have

$$\sigma^2 = \operatorname{var} W = EW^2 = \sum_{i,j} \sum_{g,h} Ew_{ij} w_{gh}. \tag{1}$$

Under *ties* among the subscripts we shall understand that one or both subscripts in the (i, j)-pair are tied with one or both subscripts in the (g, h)-pair. A *simple* tie is one in which exactly one subscript in one pair ties exactly one subscript in the other pair. Since for untied subscripts $Ew_{ij} w_{gh} = 0$, we have Var $W = O(N^3)$. The requirement Var $W = \Omega(N^3)$ implies that not all covariances between two simply tied variables w_{ij} and w_{gh} vanish.

Before we investigate the general moments of order $2k-1$ and $2k$, let us fix ideas by considering the case $k = 2$. We have

$$\mu_3 = EW^3 = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i_3, j_3} Ew_{i_1 j_1} w_{i_2 j_2} w_{i_3 j_3}.$$

$Ew_{i_1 j_1} w_{i_2 j_2} w_{i_3 j_3}$ can differ from 0 only if there are ties involving subscripts from each of the three subscript pairs. Thus $\mu_3 = O(N^4)$ and $\mu_3/\sigma^3 = O(N^4)/\Omega(N^{9/2}) = O(N^{-\frac{1}{2}})$. For μ_4 the contribution arising from ties involving three or four subscript

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