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ON THE NULL DISTRIBUTION OF THE SUM OF THE ROOTS OF A MULTIVARIATE BETA DISTRIBUTION[1]

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1. Introduction. The distribution of Pillai's V statistic [8] is shown to satisfy a homogeneous linear differential equation (d.e.) of Fuchsian type, which is related by a simple transformation to the author's d.e. for Hotelling's generalized T_0^2 [3]. This transformation implies certain relationships between the moments and asymptotic expansions of the two distributions. The adequacy of some approximations to V is checked by using the d.e. to tabulate some accurate percentage points.

2. Systems of differential equations. Let $\mathbf{S}_1$, $\mathbf{S}_2$ denote $m \times m$ matrices with independent null Wishart distributions on n_1, n_2 degrees of freedom respectively ($n_1, n_2 \geqq m$), estimating the same covariance matrix. The joint distribution of the latent roots $\theta_1, \cdots, \theta_m$ of $\mathbf{S}_1(\mathbf{S}_1+\mathbf{S}_2)^{-1}$ is well known to be

$$\phi_{n_1, n_2}(\theta_1, \cdots, \theta_m) = C(m; n_1, n_2)(\textstyle\prod_{i=1}^m \theta_i)^{\frac{1}{2}(n_1-m-1)}(\prod_{i=1}^m(1-\theta_i))^{\frac{1}{2}(n_2-m-1)} \cdot \prod_{i<j}(\theta_i-\theta_j), \qquad (0<\theta_m<\cdots<\theta_1<1), \tag{2.1}$$

where

$$C(m; n_1, n_2) = \pi^{\frac{1}{2}m^2}\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma_m(\tfrac{1}{2}m)\Gamma_m(\tfrac{1}{2}n_1)\Gamma_m(\tfrac{1}{2}n_2). \tag{2.2}$$

Pillai's V statistic is defined by

$$V = \textstyle\sum_{i=1}^m \theta_i \tag{2.3}$$

and Hotelling's generalized T_0^2 statistic by

$$T = \textstyle\sum_{i=1}^m \theta_i/(1-\theta_i) = T_0^2/n_2. \tag{2.4}$$

Following the method of [3], Section 2, we introduce the Laplace transforms (Lt's)

$$L_r(s) = \textstyle\int_{R_m} \exp(-s\sum\theta_i)\, \phi_{n_1,n_2}(\theta_1, \cdots, \theta_m)\sum_{k_1<\cdots<k_r}[(1-\theta_{k_1})\cdots(1-\theta_{k_r})]^{-1} \cdot d\theta_1\cdots d\theta_m, \qquad (r=0, 1, \cdots, m), \tag{2.5}$$

where R_m is the region defined in (2.1), and the summation is extended over the $\binom{m}{r}$ selections of r distinct integers $k_1, \cdots, k_r$ from the set $1, 2, \cdots, m$. Thus, $L_0(s)$ is the Lt of $f_{n_1,n_2}(V)$, the density function of V. For $r \geqq 1$, the integrands in (2.5) are dominated by ϕ_{n_1, n_2-2}, and so the $L_r(s)$ exist only for $n_2 \geqq m+2$. This restriction will be preserved for the present. In general, we see that

$$\textstyle\int_{R_m} \exp(-s\sum\theta_i)\psi(\theta_1, \cdots, \theta_m)\, d\theta_1 \cdots d\theta_m \tag{2.6}$$

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