

A NECESSARY CONDITION FOR GLIVENKO-CANTELLI CONVERGENCE IN E_n

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Let μ be a probability measure on the Borel sets, β , in k dimensional Euclidean space E_k and $X_1, X_2, \dots$ a sequence of independent random vectors with values in E_k such that $P(X_i \in A) = \mu(A)$ for every A in β , $i = 1, 2, \dots$. When $k \geq 2$, a necessary condition is given for

$$(1) \quad \sup_{f \in \mathcal{M}} |n^{-1} \sum_{i=1}^n f(X_i) - \int f d\mu| \rightarrow_{a.s.} 0,$$

where $\mathcal{M}$ is the class of all monotone functions on E_k . This theorem answers a conjecture of Blum [1].

For $k \geq 2$, the graph of a real-valued function with domain E_{k-1} which is strictly monotone in each variable is said to be a strictly monotone graph in E_k . Let L be the set of points x in E_k with $\mu(x) > 0$. Define $\mu^L(A)$ as $\mu(AL^c)$ for all A in β .

THEOREM. For $k \geq 2$, if there is a strictly monotone graph, B in E_k with $\mu^L(B) > 0$, then (1) does not hold.

PROOF. Suppose there is a strictly monotone graph B in E_k with $\mu^L(B) > 0$. Without loss of generality it is supposed that B is the graph of a function on E_{k-1} which is decreasing in each variable. Corresponding to $x = (x_1, x_2, \dots, x_k)$ in E_k let

$$A_x = \{(y_1, y_2, \dots, y_k) : y_i \geq x_i, i = 1, 2, \dots, k\}.$$

Let $A' = \bigcup_{x \in B} A_x$ and $A'' = A' - B$. Let f' and f'' be the characteristic functions of $A' - LB$ and A'' . Given a sample sequence, $\{x_1, x_2, \dots\}$ x_i in E_k , for each positive integer n define $A(n)$ as $\bigcup_{x_i \in A', 1 \leq i \leq n} A_{x_i}$ and f_n as the characteristic function of $A(n) - LB$. For f in $\mathcal{M}$, let $S_n(f) = n^{-1} \sum_{i=1}^n f(x_i)$. It follows that $S_n(f') = S_n(f_n)$ and that $\{f_n\}$ is a non-decreasing sequence of monotone functions on E_k . Since $A(n) - LB \subset A'' \cup [\{x_1, x_2, \dots\} \cap BL^c]$ for all n and $\mu[\{x_1, x_2, \dots\} \cap BL^c] = 0$, $\lim f_n \leq_{a.s.} f''$ and $\int \lim f_n d\mu \leq \int f'' d\mu$: Hence

$$\int f_n d\mu \rightarrow \int \lim f_n d\mu \leq \int f'' d\mu.$$

From $S_n(f') \rightarrow \int f' d\mu$ it follows that

$$\begin{aligned} \sup_{f \in \mathcal{M}} |S_n(f) - \int f d\mu| &\geq |S_n(f_n) - \int f_n d\mu| \rightarrow_{a.s.} \\ \int f' d\mu - \int \lim f_n d\mu &\geq \int f' d\mu - \int f'' d\mu = \mu^L(B). \end{aligned}$$

Hence (1) does not hold.

Define a class $\mathcal{A}$ of sets in β as follows: A is in $\mathcal{A}_1$, if $(x_1, \dots, x_k)$ in A and $y_i \leq x_i$, $i = 1, \dots, k$ imply that $(y_1, \dots, y_k)$ is in A . Let $\mathcal{A}_i$, $i = 1, 2, \dots, 2^k$ be

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