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THE HARTMAN–WINTNER LAW OF THE ITERATED LOGARITHM FOR MARTINGALES[1]

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According to the Hartman–Wintner law of the iterated logarithm [4], $\{Y_i, i \geqq 1\}$ independent identically distributed with $EY_1 = 0$ and $EY_1^2 = 1$ implies that $\limsup \sum_{i=1}^n Y_i/(2n \log_2 n)^{\frac{1}{2}} = 1$ almost surely (a.s.). We generalize this result to stationary ergodic martingale difference sequences.

THEOREM. *Let* $(Y_i, i \geqq 1)$ *be a stationary ergodic stochastic sequence with* $E[Y_i \mid Y_1, Y_2, \cdots, Y_{i-1}] = 0$ a.s. *for all* $i \geqq 2$ *and* $EY_1^2 = 1$. *Then* $\limsup \sum_{i=1}^n Y_i/(2n \log_2 n)^{\frac{1}{2}} = 1$ a.s.

PROOF. In order to prove the result, we assume a particular representation for the stochastic sequence, namely that the sample space Ω is the Cartesian product $\prod_{i=-\infty}^{\infty} R_i$ of copies of the real line, that the Y_i's are the coordinate variables of Ω, that $\mathscr{F}$ is the σ-field of events generated by the Y_i's, and that P is the probability induced on $(\Omega, \mathscr{F})$ by the finite dimensional distributions of the Y_i's. Letting $\mathscr{F}_n$ be the σ-field generated by $\{Y_m, -\infty < m \leqq n\}$, we note that $E[Y_i \mid \mathscr{F}_{i-1}] = 0$ a.s. for all integers i. Let K_i be a sequence of positive constants to be specified later and let $b_i = K_i(i/\log_2 i)^{\frac{1}{2}}$. Let $Y_i' = Y_i I(|Y_i| \leqq b_i) + b_i I(Y_i > b_i) - b_i I(Y_i < -b_i)$ and $Z_i' = Y_i' - E[Y_i' \mid \mathscr{F}_{i-1}]$. Likewise let $Y_i'' = Y_i I(|Y_i| \leqq M) + MI(Y_i > M) - MI(Y_i < -M)$ and $Z_i'' = Y_i'' - E[Y_i'' \mid \mathscr{F}_{i-1}]$ for $M > 0$. This modification of the usual truncation procedure enables us to conclude that $E[Y_i^2 \mid \mathscr{F}_{i-1}] \geqq E[(Z_i')^2 \mid \mathscr{F}_{i-1}] \geqq E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]$ a.s. when $b_i \geqq M$. (See Corollary 4 of [3] for a proof of this.) By the Birkoff ergodic theorem, $\sum_{i=1}^n E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]/n \to E(Z_0'')^2$ a.s. and $\sum_{i=1}^n E[Y_i^2 \mid \mathscr{F}_{i-1}]/n \to 1$ a.s. K_i will be chosen such that $K_i \to 0$ and $b_i \to \infty$. We then obtain $1 = \limsup \sum_{i=1}^n E[Y_i^2 \mid \mathscr{F}_{i-1}]/n \geqq \limsup \sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}]/n \geqq \limsup \sum_{i=1}^n E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]/n = E(Z_0'')^2$. Since $E(Z_0'')^2 \to 1$ as $M \to \infty$, it follows that

$$\sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}]/n \to 1 \text{ a.s.} \tag{1}$$

and hence that

$$\sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}] \to \infty \text{ a.s.} \tag{2}$$

According to [5], if $(Z_i, \mathscr{F}_i, i \geqq 1)$ is a martingale difference sequence with $s_n^2 = \sum_{i=1}^n E[Z_i^2 \mid \mathscr{F}_{i-1}] \to \infty$ a.s., $u_n = (2 \log_2 s_n^2)^{\frac{1}{2}}$, $\mathscr{F}_{i-1}$ measurable random variables $L_i \to 0$ a.s., and $|Z_i| \leqq L_i s_i/u_i$ a.s. for all $i \geqq 1$, then $\limsup \sum_{i=1}^n Z_i/(s_n u_n) = 1$ a.s.

Recalling (1) and (2), Z_i' satisfies the hypotheses of this theorem with $L_i = 2K_i u_i(i/\log_2 i)^{\frac{1}{2}}/s_i$ since $|Z_i'| \leqq 2K_i(i/\log_2 i)^{\frac{1}{2}}$ a.s. Thus, using (1), $\limsup \sum_{i=1}^n Z_i'/(2n \log_2 n)^{\frac{1}{2}} = \limsup \sum_{i=1}^n Z_i'/(s_n u_n) = 1$ a.s.

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