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AN EXTENSION OF THE HEWITT–SAVAGE ZERO-ONE LAW[1]

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Let $(\Omega^\infty, \mathfrak{A}^\infty)$ be the direct product of countably many copies of the measurable space $(\Omega, \mathfrak{A})$ and let $\mu = \prod \mu_i$ be a product probability measure on $(\Omega^\infty, \mathfrak{A}^\infty)$. The Hewitt–Savage Zero-One Law says that if all μ_i are equal then the sets of $\mathfrak{A}^\infty$ which are invariant under all permutations of finitely many coordinates have μ-measure either zero or one. We derive an extension of this theorem to a case where the μ_i are not all identical.

A product probability measure $\mu = \prod \mu_i$ is said to be *recurring* if for each $i = 1, 2, \cdots$ there is some $j > i$ such that $\mu_j = \mu_i$, i.e., each factor of μ occurs infinitely often.

THEOREM. *If $\mu = \prod \mu_i$ is recurring then $\mu(S)$ is zero or one for every set $S \in \mathfrak{A}^\infty$ which is invariant under all permutations of finitely many coordinates.*

PROOF. Let S be such a permutation invariant set. Let $\mathscr{F}_n$ be the σ-field of cylinder sets generated by the first n factors of $(\Omega^\infty, \mathfrak{A}^\infty)$. Then there exist sets $A_n \in \mathscr{F}_n$ such that $\mu(S \Delta A_n) \to 0$ as $n \to \infty$. Since μ is recurring, for each n there are n distinct indices $j_i > n$ such that $\mu_{j_i} = \mu_i$, $1 \leqq i \leqq n$. Hence for each n there exists a measure preserving transformation φ_n induced by a permutation π_n of a finite number of indices such that A_n is independent of $\varphi_n(A_n)$. Thus as $n \to \infty$

$$\mu^2(S) \leftarrow \mu(A_n)\,\mu(\varphi_n(A_n)) = \mu(A_n \cap \varphi_n(A_n)) \to \mu(S)$$

which implies $\mu(S)$ is either zero or one.

LEMMA. *Let $\mathscr{F} \supset \mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots$ be a decreasing sequence of σ-fields and let λ and ν be two probability measures on $\mathscr{F}$ whose restrictions to $\bigcap_{n=1}^\infty \mathscr{S}_n$ are identical. Then*

$$\sup_{S \in \mathscr{S}_n} |\lambda(S) - \nu(S)| \to 0 \qquad \textit{as} \quad n \to \infty.$$

PROOF. Let $\alpha \equiv \frac{1}{2}(\lambda + \nu)$ and let α_n, λ_n, ν_n be the restrictions of α, λ, ν to $\mathscr{S}_n$. Let $f_n \equiv d\lambda_n/d\alpha_n$ and $g_n \equiv d\nu_n/d\alpha_n$. Then f_n and g_n are reversed martingales which converge in $L_1(\alpha)$ to a common limit (Doob (1953)). Hence $f_n - g_n \to_{L_1(\alpha)} 0$, which implies the desired result since for $S \in \mathscr{S}_n$,

$$|\lambda(S) - \nu(S)| \leqq \|f_n - g_n\|_{L_1(\alpha)}.$$

COROLLARY. *Let μ be a recurring product probability measure on $(\Omega^\infty, \mathfrak{A}^\infty)$. Let $\mathscr{S}_n$ be the σ-field of sets in $\mathfrak{A}^\infty$ that are invariant under all permutations of the first n coordinates. If probability measures λ and ν are absolutely continuous with respect to μ, then $\sup_{S \in \mathscr{S}_n} |\lambda(S) - \nu(S)| \to 0$ as $n \to \infty$.*

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