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THE FOLDED CUBIC ASSOCIATION SCHEME

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1. The association scheme. Raghavarao and Chandrasekhararao (1964) introduced the cubic association scheme for partially balanced incomplete block designs with three associate classes. There are $v = s^3$ varieties; each variety is represented by a different set of three coordinates (z_1, z_2, z_3), where z_1, z_2, z_3 are integers and $1 \leqq z_1, z_2, z_3 \leqq s$. Two varieties are ith associates if they have $(3-i)$ coordinates the same. Thus $(0,0,0)$ has $(0,1,0)$ among its first associates and $(0,1,1)$ among its second associates.

In this paper we consider an extension of the cubic scheme for the case $s = 4$, $v = 64$. The coordinates z_1, z_2, z_3 do not take the integer values $1, 2, 3, 4$ but, instead, are elements of a Galois field of four elements: $0, 1, x$ and $y(=1+x)$, with addition modulo 2. We now introduce a fourth coordinate defined by $z_1+z_2+z_3+z_4 \equiv 0 \bmod 2$, or equivalently $z_4 \equiv z_1+z_2+z_3$, thus dividing the sixty-four points into four hyperplanes or flats. This leads to the following partially balanced scheme with four associate classes. Two varieties are first associates if two of their four coordinates are identical, e.g. $(1,0,x,y)$ and $(1,0,1,0)$, and second associates if their representations coincide at only one coordinate, such as $(0,0,0,0)$ and $(0,1,x,y)$. The three fourth associates of (z_1, z_2, z_3, z_4) are $(z_1+a,\ z_2+a,\ z_3+a,\ z_4+a)$, $a = 1, x, y$. If two varieties are not first, second, or fourth associates, they are third associates. We shall call this new scheme the folded cubic scheme. In the remainder of the paper we shall omit the commas and parentheses in denoting varieties, and write, for example, $01xy$ or 1010.

Since $z_i + z_i = 0$ for all z_i in $GF(2^2)$, it follows that the representations of the varieties fall into three types:

(i) $z_1 = z_2 = z_3 = z_4$: *aaaa*,

(ii) the z_i occur in two pairs: *aabb*, *abab*, *abba*, $a \neq b$,

(iii) the z_i are all different: *abcd*, $a \neq b \neq c \neq d$.

Thus two varieties cannot have more than two coordinates equal, and $n_1 = 18$, $n_2 = 24$, $n_3 = 18$, $n_4 = 3$.

The fourth associates of *aaaa* are *bbbb*, *cccc*, *dddd*; for *aabb* the fourth associates are *bbaa*, *ccdd*, *ddcc*; for *abcd* they are *badc*, *cdab* and *dcba*.

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