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SUPERPOSITIONS AND THE CONTINUITY OF INFINITELY DIVISIBLE DISTRIBUTIONS

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1. Basic properties of infinitely divisible distributions. Let $F_Y(x) = P[Y \leqq x]$ be an infinitely divisible probability distribution function. Its characteristic function $f_Y(u) = Ee^{iuY}$ has a representation of the form

$$f_Y(u) = \exp\left\{i\gamma_Y u - \frac{\sigma_Y{}^2 u^2}{2} + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dM_Y(x)\right\},$$

where $\gamma_Y, \sigma_Y{}^2$, and M_Y are the Lévy parameters uniquely associated with the given distribution. The Lévy spectral function M_Y is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$, is asymptotically zero $(M(-\infty) = 0 = M(\infty))$, and satisfies the integrability condition

$$\int_{-1}^{-0} + \int_{+0}^{+1} x^2\, dM_Y(x) < \infty.$$

In [2], P. Hartman and A. Wintner proved that a necessary and sufficient condition that F_Y be continuous is that $\int_{-\infty}^{\infty} dM_Y(x) = \infty$ or $\sigma_Y{}^2 > 0$. Howard G. Tucker in [4] and M. Fisz and V. S. Varadarajan in [1] have shown that a sufficient condition for F_Y to be absolutely continuous is that $\int_{-\infty}^{\infty} dM_{ac}(x) = \infty$, where M_{ac} denotes the absolutely continuous component of M_Y.

We shall show that the distribution functions associated with certain non-negative infinitely divisible random variables are continuous if and only if they are continuous at their first rise.

2. Superposition of Brownian motion onto an infinitely divisible random variable. Let $\{W(t)/t \in [0, \infty)\}$ be a standard Brownian motion; i.e. a separable differential process with sample paths that are almost surely continuous and such that $\mathscr{L}(W(t)) = \mathscr{N}(0, t)$. Let X be a nonnegative infinitely divisible random variable that is independent of $\{W(t)\}$. The Lévy spectral function M_X associated with X vanishes on the negative half-axis and we assume that it satisfies the stronger integrability condition

$$\int_{+0}^{+1} x\, dM_X(x) < \infty. \tag{1}$$

Consequently, the characteristic function of X can be written in the form

$$f_X(u) = \exp\{i\gamma_X u + \int_0^{\infty} (e^{iux} - 1)\, dM_X(x)\},$$

where $\gamma_X \geqq 0$. The superposition $Y = W(X)$ also has an infinitely divisible distribution. We have shown in [3] (more general results appear in [3] and [5]) that $Y = W(X)$ has Lévy parameters $\gamma_Y = 0$, $\sigma_Y{}^2 = \gamma_X$, and

$$\begin{aligned} M_Y(x) &= \int_{-\infty}^{x} \int_0^{\infty} (2\pi t)^{-\frac{1}{2}} \exp\{-y^2/2t\}\, dM_X(t)\, dy, && x < 0 \\ &= -M_Y(-x), && x > 0. \end{aligned}$$

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