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ON THE L_p-CONVERGENCE FOR $n^{-1/p}S_n$, $0 < p < 2$[1]

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Let $(X_n, n \geqq 1)$ be a sequence of random variables and $S_n = X_1 + \cdots + X_n$. By an ingenious method, Pyke and Root [4] prove that if $X_1, X_2, \cdots$ are i.i.d. random variables with $E|X_1|^p < \infty$ for some $0 < p < 2$, then $E|S_n - a_n|^p = o(n)$ as $n \to \infty$, where $a_n = 0$ if $0 < p < 1$ and $a_n = nEX_1$ if $1 \leqq p < 2$. By using an inequality due to Essen and Von Bahr [3], Chatterji [2] extends the result to the following form: If $X_1, X_2, \cdots$ are dominated in distribution by a random variable X with $E|X|^p < \infty$ for some $0 < p < 2$, then $E|S_n - a_n|^p = o(n)$ as $n \to \infty$, where $a_n = 0$ if $0 < p < 1$ and $a_n = \sum_1^n E(X_k | X_1, \cdots, X_{k-1})$ if $1 \leqq p < 2$. In this note, by applying an inequality due to Burkholder [1], we will prove the following result, which relaxes the domination condition of [2] to uniform integrability.

THEOREM. *Let* $(|X_n|^p, n \geqq 1)$ *be uniformly integrable for some* $0 < p < 2$. *Then as* $n \to \infty$

$$E|S_n - a_n|^p = o(n),$$

where $a_n = 0$ *if* $0 < p < 1$, *and* $a_n = \sum_1^n E(X_k | X_1, \cdots, X_{k-1})$ *if* $1 \leqq p < 2$.

PROOF. Define $Y_k = X_k$ if $0 < p < 1$ and $Y_k = X_k - E(X_k | X_1, \cdots, X_{k-1})$ for $1 \leqq p < 2$. It is easy to see that if $(|X_n|^p, n \geqq 1)$ is uniformly integrable, so is $(E(|X_n|^p | X_1, \cdots, X_{k-1}), n \geqq 1)$. Hence $(|Y_n|^p, n \geqq 1)$ is uniformly integrable. For $\varepsilon > 0$, choose $M > 0$ so that $\int_{[|Y_n| > M]} |Y_n|^p < \varepsilon$ for all $n \geqq 1$. Put

$$Y_n' = Y_n I_{[|Y_n| \leqq M]}, \qquad Y_n'' = Y_n - Y_n'.$$

(a) If $0 < p < 1$,

$$\begin{aligned} E|S_n|^p &= E\left|\sum_1^n (Y_k' + Y_k'')\right|^p \leqq E\left(\sum_1^n |Y_k'|\right)^p + E\left(\sum_1^n |Y_k''|\right)^p \\ &\leqq E\left(\sum_1^n |Y_k'|\right)^p + \sum_1^n E|Y_k''|^p \leqq (nM)^p + n\varepsilon, \end{aligned}$$

and hence $E|S_n|^p = o(n)$ as $n \to \infty$.

(b) If $1 < p < 2$, by Burkholder inequality [1] there exists a constant $A_p > 0$ satisfying

$$A_p E\left|\sum_1^n Y_k\right|^p \leqq E\left(\sum_1^n Y_k^2\right)^{p/2}, \qquad n \geqq 1.$$

Hence

$$\begin{aligned} A_p E|S_n - a_n|^p &= A_p E\left|\sum_1^n Y_k\right|^p \leqq E\left(\sum_1^n Y_k^2\right)^{p/2} \\ &= E\left\{\sum_1^n (Y_k'^2 + Y_k''^2)\right\}^{p/2} \leqq E\left(\sum_1^n Y_k'^2\right)^{p/2} + E\left(\sum_1^n Y_k''^2\right)^{p/2} \\ &\leqq E\left(\sum_1^n Y_k'^2\right)^{p/2} + E\sum_1^n |Y_k''|^p \leqq (nM^2)^{p/2} + n\varepsilon. \end{aligned}$$

Therefore $E|S_n - a_n|^p = o(n)$ as $n \to \infty$.

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