The Annals of Mathematical Statistics
1971, Vol. 42, No. 1, 376–380

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR k-DIMENSIONAL RENEWAL THEORY[1]

BY DOUGLAS P. KENNEDY

Stanford University[2]

1. Introduction. Let $\{\mathbf{X}_n, n \geqq 1\}$ be a sequence of random vectors in $\mathbb{R}^k$ defined on some probability triple $(\Omega, \mathscr{F}, P)$ and set $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_i$ for $n \geqq 1$, and $\mathbf{S}_0 = \mathbf{0}$. Let $h: \mathbb{R}^k \to [0, \infty)$ be a function with continuous first partial derivatives, such that $h(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, $\mathbf{x} \in \mathbb{R}^k$; assume furthermore that h is homogeneous of degree one (i.e., for all $\mathbf{x} \in \mathbb{R}^k$, $\lambda \geqq 0$, $h(\lambda\mathbf{x}) = \lambda h(\mathbf{x})$). We define the associated point process $\{M(t): t \geqq 0\}$ by $M(t) = \min\{n \geqq 1 : h(\mathbf{S}_n) > t\}$, where $M(t) = \infty$ if no such n exists.

The main result of this paper is a functional central limit theorem (invariance principle) for the process $\{M(t): t \geqq 0\}$. Section 2 is devoted to two preliminary lemmas and the theorem is proved in Section 3.

The ordinary central limit theorem for $\{M(t): t \geqq 0\}$ was given by Farrell [4]. Bickel and Yahav [1] discuss renewal theory for which h is any norm giving the Euclidean topology in $\mathbb{R}^k$. Related material on k-dimensional renewal theory may be found in Farrell [3] and Stam [5].

Our analysis shall be carried out in $D[0, 1]$, the space of right continuous functions on $[0, 1]$ having left limits and endowed with the Skorohod metric d. For an account of the weak convergence of probability measures on $D[0, 1]$ the reader is referred to the book by Billingsley (1968). We shall use $\Rightarrow$ to denote weak convergence of probability measures. When stochastic processes or ordinary random variables appear in such an expression we mean the measures induced by these functions. Let $C[0, 1] \equiv C$ denote the space of continuous functions on $[0, 1]$ and ρ the uniform metric on C and D; $C^k \equiv C^k[0, 1]$ and $D^k \equiv D^k[0, 1]$ will denote the product spaces of k copies of C and D respectively, with the appropriate product topologies.

2. Preliminaries. Let $\boldsymbol{\mu} \in \mathbb{R}^k$, $\boldsymbol{\mu} \neq \mathbf{0}$ and define the random functions Y_n, H_n in D^k and D induced by the sequence of partial sums $\{\mathbf{S}_n, n \geqq 1\}$ as follows

$$Y_n(t) = [\mathbf{S}_{[nt]} - nt\boldsymbol{\mu}]/n^{\frac{1}{2}}$$
$$H_n(t) = [h(\mathbf{S}_{[nt]}) - nth(\boldsymbol{\mu})]/n^{\frac{1}{2}}.$$

Let $\cdot$ denote the ordinary scalar product in $\mathbb{R}^k$ and $\nabla h = (\partial h/\partial x_1, \cdots, \partial h/\partial x_k)$. Note that ∇h is a homogeneous function of degree 0, in particular $\nabla h(t\boldsymbol{\mu}) = \nabla h(\boldsymbol{\mu})$ for all $t \in [0, 1]$.

LEMMA 1. *If $Y_n \Rightarrow \xi$ in D^k then $H_n \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot \xi$ in D, where $\nabla h(\boldsymbol{\mu}) \cdot \xi$ is the scalar product of the process ξ and the constant vector $\nabla h(\boldsymbol{\mu})$.*

Received February 12, 1970. Author now at the University of Sheffield, England.

[1] This work was supported by Office of Naval Research Contract N00014-67-A-0112-0031 and National Science Foundation Grant GP-8790.

[2] Now at the University of Sheffield, England