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NOTE ON THE CHARACTERIZATION OF CERTAIN ASSOCIATION SCHEMES

BY MARTIN AIGNER

University of North Carolina

1. Introduction. Much attention has been paid in recent years to characterizing association schemes by means of their parameters (see e.g. [2]). There have been essentially two different approaches, one using the concept of a claw, e.g. [2, 3], the other considering the eigenvalues of the corresponding strongly regular graph, e.g. [4]. In this note we suggest another method, namely, to study the structure of the set of treatments which are first associates of a given treatment, using all the information contained in the parameters n_1, p_{11}^1, p_{11}^2. It appears convenient to discuss the problems in terms of the strongly regular graph obtained from the association scheme by joining two treatments iff they are first associates. Henceforth we shall adopt the graphtheoretic language. To give a concrete example let us assume the graphs satisfy $p_{11}^2 \leqq 2$. This now implies the subgraph $A(x)$ generated by all points adjacent to a given point x does not contain a cycle of length four unless it is embedded in a complete graph on four points. This requirement considerably restricts the class of possible graphs $A(x)$, and in certain cases readily yields the solution of the characterization problem. To give an application we shall exhibit the characterization of the line graph of the complete bipartite graph [5], [6], [7]. The advantage of the present approach is that it lends itself to generalization to several associate classes (see [1]), admits variable p_{11}^1, p_{11}^2 and also produces all the exceptions.

2. A class of graphs. It is our goal to characterize the line graph of the complete bipartite graph on sets with m and n vertices, denoted by $L(B_{m,n})$, by the following properties:

For $m \geqq n \geqq 2$

(P1) $L(B_{m,n})$ has $m \cdot n$ vertices.

(P2) It is regular of degree $m+n-2$.

(P3) Exactly $n \cdot \binom{m}{2}$ pairs of adjacent points are mutually adjacent to $m-2$ points, the remaining $m \cdot \binom{n}{2}$ pairs of adjacent points are mutually adjacent to $n-2$ points.

(P4) Any two nonadjacent points are mutually adjacent to two points.

We now define for $m \geqq n \geqq 2$ the following class $\mathscr{G}(m, n)$ of graphs G:

(Q1) G contains $m+n-2$ points.[1]

(Q2) The degree of a point in G is either $m-2$ or $n-2$, but there are at least $m-1$ points of degree $m-2$.

(Q3) There is no cycle of length 4 in G unless it is embedded in a complete graph on 4 points.

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[1] It may be noted that (Q1)–(Q3) and thus the following theorem are also applicable to the line graph of a BIB design with m replications and block size n.