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NOTES

A NOTE ON THE MULTIVARIATE t-RATIO DISTRIBUTION[1]

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1. Introduction. Press [2] has studied the distribution of the ratio of two random variables which follow the bivariate t-distribution. He has presented the density function of a linear function of the ratio. It is the purpose of this paper to extend these results to the multivariate case. That is to say, suppose that the random vector X, where $X' = (x_1, x_2, \cdots, x_p)$, has the multivariate Student t density function

$$f(x; \theta, \tau) = c[\nu + (X-\theta)'\tau(X-\theta)]^{-(\nu+p)/2}$$

where

$$c = \pi^{-p/2}|\tau|^{\frac{1}{2}}\nu^{\nu/2}\Gamma[\tfrac{1}{2}(\nu+p)]/\Gamma[\tfrac{1}{2}\nu].$$

We shall find the density function of the random vector Y, where $Y' = (y_1, y_2, \cdots, y_{p-1}) = (x_2/x_1, x_3/x_1, \cdots, x_p/x_1)$.

2. The density function when p is odd. Let $z = x_1$, $y_1 = x_2/x_1$, $y_2 = x_3/x_1, \cdots, y_{p-1} = x_p/x_1$. The Jacobian of this transformation is z^{p-1}. Also, $X = zW$, where $W' = (1, y_1, y_2, \cdots, y_{p-1})$. The joint density function of Y and z is

$$h(z, Y) = c[\nu + (zW-\theta)'\tau(zW-\theta)]^{-(\nu+p)/2}z^{p-1}$$

and the density function of Y is

$$g(Y) = cM^{-(\nu+p)/2}\int_{-\infty}^{\infty}[z^2 + (K/M)z + (L/M)]^{-(\nu+p)/2}z^{p-1}\,dz$$

where $M = W'\tau W$, $K = -2W'\tau\theta$, and $L = \nu + \theta'\tau\theta$.

Now $a = (L/M) - (K^2/4M^2)$ is always positive for $(L/M)-(K^2/4M^2) = \nu(W'\tau W)^{-1} + (W'\tau W)^{-2}[(W'\tau W)(\theta'\tau\theta) - (W'\tau\theta)^2]$. But τ is positive definite and $(W'\tau W)(\theta'\tau\theta) - (W'\tau\theta)^2 \geqq 0$ (see Rao [3] page 43).

Let $u = (z-b)/a$, where $b = -K/2M$. Then

$$\begin{aligned} g(Y) &= c(Ma)^{-(\nu+p)/2}a\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}[au+b]^{p-1}\,du \\ &= c(Ma)^{-(\nu+p)/2}a\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}(au)^{p-1-i}b^i\,du. \end{aligned}$$

Since p is an odd integer

$$\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}u^{p-2j}\,du = 0$$

for $j = 1, 2, \cdots, (p-1)/2$.

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