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INFINITESIMAL LOOK-AHEAD STOPPING RULES[1]

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1. Introduction. Let $X = (X_t,\ t \geqq 0)$ be a strong Markov Process having stationary transition distributions, and sample paths which are almost surely right continuous and have only jump discontinuities. The state space S of the process is assumed to be a Borel subset of a complete separable metric space and we consider the problem of selecting a stopping time τ maximizing

$$(1) \qquad E^x[e^{-\lambda\tau} f(X_\tau) - \int_0^\tau e^{-\lambda s} c(X_s)\, ds],$$

where f and c are continuous real-valued functions on S, $\lambda \geqq 0$, and E^x denotes expectation conditional on $X_0 = x$.

In the second section of this paper, we show that under certain conditions an infinitesimal look-ahead procedure is optimal. This result generalizes certain discrete time results given by Derman–Sacks (1960) in [5] and independently by Chow–Robbins (1961) in [4]. In the third section, a related approach is described and the resultant procedure is shown to be optimal under slightly more general situations. The fourth section considers a class of continuous time Markovian Decision Processes for which the criterion function is closely related to (1).

2. Infinitesimal look-ahead stopping rule. A stopping time τ is defined to be any nonnegative extended real-valued random variable such that for all $t > 0$, $\{\tau \leqq t\}$ is contained in the sigma field generated by $\{X_s,\ 0 \leqq s \leqq t\}$. A stopping time τ^* is said to be optimal at $x \in S$ if

$$E^x[e^{-\lambda\tau^*} f(X_{\tau^*}) - \int_0^{\tau^*} e^{-\lambda s} c(X_s)\, ds] = \max_\tau E^x[e^{-\lambda\tau} f(X_\tau) - \int_0^\tau e^{-\lambda s} c(X_s)\, ds].$$

If τ^* is optimal at x for every $x \in S$, then it is said to be optimal.

Define the infinitesimal operator $\alpha(x)$ by

$$(2) \qquad \alpha(x) = \lim_{h \to 0^+} E^x\left[\frac{f(X_h) - f(x)}{h}\right], \qquad x \in S.$$

We assume that f and X are such that the limit in (2) exists.

We first state the following well-known result. For a proof, the reader should consult Breiman [3], page 376.

LEMMA 2.1. *Suppose that both f and α are bounded and continuous.*
(a) *For any stopping time τ and $\lambda > 0$,*

$$E^x[e^{-\lambda\tau} f(X_\tau)] - f(x) = E^x[\int_0^\tau e^{-\lambda s}(\alpha(X_s) - \lambda f(X_s))\, ds].$$

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