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RANDOM TIME TRANSFORMATIONS OF SEMI-MARKOV PROCESSES

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1. Introduction. Let $\{X_t : t \geqq 0\}$ be a right continuous stochastic process taking values in some space $(E, \mathscr{E})$ and adapted to a right continuous family of σ-algebras $\{\mathscr{F}_t : t \geqq 0\}$, and let $\{\tau_t : t \geqq 0\}$ be a nonnegative real-valued right continuous stochastic process such that each τ_t is an $\mathscr{F}_t$-stopping time. Then the process $X_t' = X(\tau_t)\ t \geqq 0$, which is adapted to $\{\mathscr{F}_{\tau_t}\}$, is called the random time transformation (RTT) of $\{X_t\}$ determined by the change of time $\{\tau_t\}$.

Two classical RTT's that transform Markov processes into Markov processes are found in Dynkin [3] (or in Blumenthal and Getoor [2]) and in Feller [4]. For the RTT in Dynkin, $\{\tau_t\}$ is the inverse mapping of a strictly increasing continuous additive functional of $\{X_t\}$. This RTT is used in constructing generalized Brownian motion from a Wiener process. Feller discusses RTT's under the heading of subordination of processes. He shows that if $\{X_t\}$ is a Markov process with continuous transition probabilities and $\{\tau_t\}$ has stationary, independent, nonnegative increments and is independent of $\{X_t\}$, then $\{X_t'\}$ is a Markov process. This type of RTT was used by Bochner to construct symmetric, stable processes from a Wiener process. In addition to their theoretical significance, RTT's are useful in applications where $\{X_u : u \geqq 0\}$ describes some phenomena as a function of some parameter u, which increases in time according to a process $\{\tau_t\}$, and one is interested in $\{X_t'\}$ which depicts the phenomena as a function of time.

In this paper we present several RTT's of semi-Markov step processes (SMP's) (Definition 2). In Theorem 1, our major result, we identify some general conditions on a time process $\{\tau_t\}$ under which $\{X_t'\}$ is an SMP whenever $\{X_t\}$ is an SMP. We use this result in Section 4 to identify four types of RTT's. Two of these RTT's are analogous to those discussed by Dynkin and Feller for Markov processes, and a special case of another is similar to the RTT presented by Yackel [15]. Special cases of the RTT's presented transform Markov chains or regular step Markov processes or SMP's into any one of these three classes of processes. We conclude in Section 5 by presenting two other RTT's for special SMP's $\{X_t\}$ when $\{\tau_t\}$ is a step process independent of $\{X_t\}$.

2. Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a probability space: all stochastic processes introduced herein will be defined on this space. Let E be a locally compact Hausdorff space with a countable base, and set $\mathscr{E} = \mathscr{B}(E)$, the smallest σ-algebra containing the Borel sets of E. Let $R_+ = [0, \infty)$ and set $\mathscr{B}_+ = \mathscr{B}(R_+)$.

Let $Q(x, G)$ be a real-valued function defined for each $x \in E$ and $G \in \mathscr{E} \times \mathscr{B}_+$ with the properties:

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