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# PROCESSES OBTAINABLE FROM BROWNIAN MOTION BY MEANS OF A RANDOM TIME CHANGE[1,2]

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**1. Introduction.** Our terminology throughout this paper will in general be that of [8]. The triple $(\Omega, \mathscr{A}, P)$ will denote our fixed fundamental probability space. A random variable will be an $\mathscr{A}$-measurable real-valued function. Throughout this paper we will assume that the Brownian motion we deal with is standard Brownian motion with *all* sample paths continuous and unbounded in both directions. If $\{X(t): t \in [a, b]\}$ is a collection of random variables, then $\sigma\{X(t): t \in [a, b]\}$ will denote the smallest sub-sigma field of $\mathscr{A}$ for which each $X(t)$, $t \in [a, b]$, is measurable. Furthermore if $X$ is a random variable and $A \in \mathscr{A}$ then $[X \leqq s]$ will denote the event $\{\omega \in \Omega: X(\omega) \leqq s\}$ and $I_A$ will denote the indicator of the event $A$.

The problem of finding what processes are random time changes of Brownian motion has been studied extensively (in the case of martingales) by K. E. Dambis in [2], by L. E. Dubins and G. Schwarz in [4]. In [4] L. E. Dubins and G. Schwarz showed that every continuous martingale can be transformed into standard Brownian motion by means of a random time change. In this paper we prove that if $\{X(t): t \in [0, +\infty)\}$ is a Brownian motion process and if $\{Y(\alpha): \alpha \in I\}$ is a stochastic process with sufficiently nice properties then $\{Y(\alpha): \alpha \in I\}$ can be obtained from $\{X(t): t \in [0, +\infty)\}$ by means of a random time change (see Definition 2.1 below). Furthermore for certain processes $\{Y(\alpha): \alpha \in [0, +\infty)\}$, the collection of stopping times we construct, "almost" has independent increments. The main results of this paper are Theorem 2.2, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Corollary 2.7.

**2. Main results.**

DEFINITION 2.1. Let $I \subset [0, +\infty)$ and let $\{X(t): t \in [0, +\infty)\}$ and $\{Y(\alpha): \alpha \in I\}$ be stochastic processes defined on $(\Omega, \mathscr{A}, P)$. Then we say that $\{Y(\alpha): \alpha \in I\}$ can be obtained from $\{X(t): t \in [0, +\infty)\}$ by means of a random time change if and only if there exists a collection of random variables $\{T_\alpha: \alpha \in I\}$ defined on $(\Omega, \mathscr{A}, P)$ satisfying the following requirements:

$$\text{for each} \quad \alpha \in I, \qquad T_\alpha \geqq 0, \tag{2.1}$$

$$\text{for each} \quad \omega \in \Omega, \qquad T_\alpha(\omega) \text{ is non-decreasing in } \alpha, \tag{2.2}$$

$$\text{for each} \quad \alpha \in I, [T_\alpha \leqq s] \in \sigma\{X(t): t \in [0, s]\} \quad \text{for every} \quad s \in [0, +\infty), \quad \text{and} \tag{2.3}$$

$$\text{for each} \quad \alpha \in I, \qquad X(T_\alpha) = Y(\alpha) \quad \text{a.s.} \tag{2.4}$$

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