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# OPTIMAL DESIGNS FOR MULTIVARIATE POLYNOMIAL EXTRAPOLATION[1]

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**1. Introduction.** Let $f = (f_1, f_2, \cdots, f_k)$ be a vector of linearly independent continuous functions on a compact set $X$ in Euclidean $m$-space. For each "level" $x$ in $X$ an experiment can be performed whose outcome is a random variable $Y(x)$ with mean value $\sum_{i=1}^{k} \theta_i f_i(x)$ and variance $\sigma^2$, independent of $x$. The functions $f_1, f_2, \cdots, f_k$ are called the regression functions and assumed known to the experimenter, while the vector of parameters $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$ and $\sigma^2$ are unknown. We will be concerned here with the problem of estimating the regression function $\sum_{i=1}^{k} \theta_i f_i(\bar{x})$, at a point $\bar{x}$ outside of $X$, by means of a finite number of uncorrelated observations $Y(x_i)$. The design problem is one of selecting the levels $x_i$ in $X$ at which to experiment. The result here is approximate in that we consider a design to be an arbitrary probability measure on $X$. For a more complete discussion of the model see Kiefer (1959) or Karlin and Studden (1966).

For the case $X = [-1, 1]$ and $\sum_{i=1}^{k} \theta_i f_i(x) = \sum_{i=1}^{k} \theta_i x^{i-1}$, Hoel and Levine (1964) showed that the optimum design for estimating $\sum_{i=1}^{k} \theta_i \bar{x}^{i-1}$ (for any $\bar{x} \notin [-1, 1]$) was supported on the points $x_\nu = -\cos \nu\pi/(k-1)$, $\nu = 0, 1, \cdots, k-1$. Kiefer and Wolfowitz (1965), Studden (1968) and Studden and Karlin (1966) give further results for the case where the system $\{f_i(x)\}_1{}^k$ is a Tchebycheff system. Hoel (1965) gives a discussion of the extrapolation problem in multidimensions when the regression function is essentially of a product type.

In the present paper we consider the case where the regression function is a polynomial in $m$ dimensions of degree less than or equal to $n$. The domain $X$ will be a compact *convex* subset of the Euclidean $m$-space. Thus we take our $f_i$ to be the functions $x_1{}^{\alpha_1} \cdots x_m{}^{\alpha_m}$ where the $\alpha_j$ are nonnegative integers and $\sum_{j=1}^{m} \alpha_j \leqq n$. The number of such functions is $k = \binom{n+m}{m}$ and we assume that they are arranged in some fixed order.

**2. Optimal design.** The optimal extrapolating design is described as follows. Consider a line through $\bar{x}$ which intersects the convex set $X$ at two points, say $a$ and $b$, such that the tangent hyperplanes at $a$ and $b$ are parallel. (The line in question exists but is not necessarily unique). The optimal design for extrapolating to $\bar{x}$ is now obtained by using the one-dimensional result for polynomials of degree $n$ on the line through $a$ and $b$. Thus we consider the transformation $x(\alpha) = [(1-\alpha)a + (1+\alpha)b]/2$, such that $x(-1) = a$ and $x(+1) = b$. The optimal design concentrates on the points $x_\nu = x(\alpha_\nu)$ where $\alpha_\nu = -\cos \nu\pi/n$, $\nu = 0, 1, \cdots, n$. The optimal weights $p_\nu$, $\nu = 0, 1, \cdots, n$ can be found as in the one-dimensional case

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