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INADMISSIBILITY OF THE BEST INVARIANT TEST IN THREE OR MORE DIMENSIONS

BY STEPHEN PORTNOY[1] AND CHARLES STEIN[2]

Harvard University

Consider the problem where one observes (X, Y) with X an arbitrary random quantity and Y a p-dimensional random vector, and where there are two hypotheses, H_i, under which $(X, Y-\eta)$ is distributed according to P_i $(i = 0, 1)$ for some unknown point $\eta \in R^P$. Lehmann and Stein [1] show that if $p = 1$, then under certain reasonable conditions, the best invariant test of H_0 versus H_1 is admissible. Here we present an example showing that the analogous result does not hold if $p \geqq 3$. The example was suggested by certain problems in the recovery of interblock information in the randomized designs considered in [2]. Although the present example is somewhat artificial and not directly applicable to the above problems, it is not unlikely that similar methods might also work there.

Let η be an unknown point in R^P and let $\varepsilon_i = (-1)^i$ for $i = 0, 1$. Suppose $X = (W, V)$ and Y are distributed as follows under H_i: W is normally distributed with mean ε_i and variance 1, V is independent of W and uniformly distributed over the surface of the unit sphere in R^P, and $Y = \eta + \varepsilon_i V$. The problem is invariant under translation: $(W, V, Y) \to (W, V, Y+c)$ for $c \in R^P$. An invariant test is one depending only on (W, V) and the best invariant test of given size accepts H_0 if and only if $W > K$. We shall take $K = 0$ for ease of computation. We want to show that for sufficiently small $a > 0$ and sufficiently large b, we have for all η

$$P_{0,\eta}\left\{W + \frac{aV'Y}{b+||Y||^2} > 0\right\} > P_0\{W > 0\}, \tag{1}$$

$$P_{1,\eta}\left\{W + \frac{aV'Y}{b+||Y||^2} > 0\right\} < P_1\{W > 0\}, \tag{2}$$

where V and Y are column vectors, V' denotes the transpose of V and $||Y||^2 = Y'Y$. Because of the symmetry of the problem under interchange of the two hypotheses, we need only prove (1).

We use the identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{B}{A^2} + \frac{B^2}{A^2(A+B)} \tag{3}$$

with

$$A = b + ||\eta||^2, \qquad B = 2\eta'V + 1, \tag{4}$$

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