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A NOTE ON CONVERGENCE OF MOMENTS

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In this note, some results in [4] on convergence of even-integer moments in a central limit situation are extended in Theorem B, Section 2 to cover the case of non-even-integer absolute moments. In Section 1 we note that related results of the author ([2], [3]) were first proved by S. N. Bernstein [1], a long time ago.

1. Results of Bernstein. Let $X_1, X_2, \cdots$ be independent random variables (rv's) with $EX_n = 0, EX_n^2 = \sigma_n^2 < \infty, S_n = X_1 + \cdots + X_n$, and $s_n^2 = ES_n^2$, for $n = 1, 2, \cdots$. Among the results in [2] and [3] is the following

THEOREM A. *For each* $v > 2$, *the condition*

$$\lim_{n\to\infty} s_n^{-v} \sum_{j=1}^n E|X_j|^v = 0 \tag{1}$$

is necessary and sufficient for

$$\mathscr{L}(S_n/s_n) \to N(0, 1) \quad \text{as} \quad n \to \infty,$$

$$\lim_{n\to\infty} s_n^{-2} \max_{j \leqq n} \sigma_j^2 = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} E|S_n/s_n|^v = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |x|^v \exp(-\tfrac{1}{2}x^2)\, dx.$$

Dr. G. K. Eagleson has pointed out that this theorem is by no means new; in fact it was proved by S. N. Bernstein [1] fully thirty years ago, using symmetrization methods which do not rely on characteristic functions (ch.f's) as in [3]. The condition (1), called a *Lindeberg* condition of order v in [2] and [3], was given the possibly more apt name of "convergence to zero of the *Liapounov* parameter of order v," by Bernstein in [1]. The question of nomenclature is avoided in the present work by referring to such conditions simply as "L_v".

2. Convergence of moments. Following the notation of [4], let $X_{n1}, X_{n2}, \cdots, X_{nj_n}$, $n = 1, 2, \cdots$ be an elementary system of zero mean independent rv's, i.e., each X_{nj} has mean zero, distribution function $F_{nj}(\cdot)$, and variance DX_{nj}, with

$$\lim_{n\to\infty} \max_j DX_{nj} = 0, \qquad \text{and}$$

$$\sum_j DX_{nj} \leqq \text{ some } C < \infty \qquad \text{for all } n = 1, 2, \cdots. \tag{2}$$

Let $S_n = \sum_j X_{nj}$. We assume throughout that S_n converges in law as $n \to \infty$ to an infinitely divisible rv. T_0 with ch.f. $\phi_0(\cdot)$ given by

$$\log \phi_0(t) = \int_{-\infty}^{\infty} (e^{itx} - 1 - itx) x^{-2}\, dG_0(x),$$

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