The Annals of Mathematical Statistics
1971, Vol. 42, No. 3, 1088-1090

INFINITE DIVISIBILITY OF DISCRETE DISTRIBUTIONS, II

BY W. D. WARDE[1] AND S. K. KATTI[2]
Florida State University

1. Introduction. Let $\{P_i\}$, $i = 0, 1, 2 \cdots, P_0 \neq 0, P_1 \neq 0$ represent the probability of a random variable X taking on the values $0, 1, 2, \cdots$. Katti (1967) shows that a necessary and sufficient set of conditions for X to be infinitely divisible is that

$$\pi_i = iP_i/P_0 - \textstyle\sum_{j=1}^{i-1} \pi_{i-j} P_j/P_0 \geqq 0 \tag{1}$$

for $i = 1, 2, \cdots$. The aim of this paper is to derive a sufficient condition and to present some additional results.

2. A sufficient condition.

THEOREM 2.1. *A discrete distribution* $\{P_i\}$ $i = 0, 1, \cdots, P_0 \neq 0, P_1 \neq 0$ *is infinitely divisible if* $\{P_i/P_{i-1}\}$ $i = 1, 2, \cdots$ *forms a monotone increasing sequence.*

PROOF. Denote P_i/P_{i-1} by K_i. By assumption, $K_1 \leqq K_2 \leqq \cdots$. Note that since P_1 and P_0 are nonzero, $\{K_i\}$ $i = 1, 2, \cdots$ are positive and nonzero. Now,

$$\frac{P_i}{P_0} = \frac{P_i}{P_{i-1}} \frac{P_{i-1}}{P_{i-2}} \cdots \frac{P_1}{P_0} = \prod_{j=1}^{i} K_j. \tag{2}$$

From (1),

$$\pi_i = \frac{iP_i}{P_0} - \sum_{j=1}^{i-1} \frac{P_j \pi_{i-j}}{P_0} = i\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=1}^{i-1} \pi_{i-j}\left(\prod_{k=1}^{j} K_k\right). \tag{3}$$

On replacing i by $(i+1)$ in (3), we get

$$\pi_{i+1} = (i+1)\left(\prod_{j=1}^{i+1} K_j\right) - \sum_{j=0}^{i-1} \pi_{i-j}\left(\prod_{k=1}^{j+1} K_k\right). \tag{4}$$

Hence,

$$\frac{\pi_{i+1}}{K_{i+1}} = (i+1)\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=0}^{i-1} \frac{\pi_{i-j}}{K_{i+1}}\left(\prod_{k=1}^{j+1} K_k\right). \tag{5}$$

On using property (2), we get,

$$\frac{\pi_{i+1}}{K_{i+1}} \geqq i\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=1}^{i-1} \pi_{i-j}\left(\prod_{k=1}^{j} K_k\right) - \frac{\pi_i K_1}{K_{i+1}}, \tag{6}$$

Received May 9, 1968.

[1] Now at Iowa State University.

[2] Now at University of Missouri.

Research supported by the United States Department of Agriculture, Grant 12-14-100-9183 (20).