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ON A THEOREM OF G. L. SIEVERS

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In [3] Sievers proved a rather general theorem on the probability of large deviations. In this note a simpler method of proof is used to show that this theorem holds under weaker assumptions than those made by Sievers. In particular, no assumptions on the rate of convergence of the underlying sequences are necessary.

THEOREM. Let $\{W_n\}_{n=1,2,\dots}$ *be a sequence of random variables on a probability space* $(\Omega, \mathscr{A}, P)$ *which satisfies the following assumptions*:

(i) $m_n(t) = \int e^{tW_n}\,dP < \infty$, $t \in [0, T)$, $T > 0$.

(ii) $\lim_{n\to\infty} n^{-1}\Psi_n^{(k)}(t) = c_k(t) < \infty$, $t \in [0, T)$, $k = 0, 1, 2$, *where* $\Psi_n(t) = \ln m_n(t)$.

(iii) $c_2(t) > 0$, $t \in [0, T)$.

(iv) $n^{-1}\Psi_n^{(3)}(t)$ *is locally bounded on* $(0, T)$. *Then for any sequence* $\{a_n\}_{n=1,2,\dots}$ *of real numbers with* $\lim_{n\to\infty} a_n = a \in \{c_1(t) : t \in (0, T)\}$ *it holds that*

$$\lim_{n\to\infty} [P(W_n > na_n)]^{1/n} = \exp[c_0(h) - ha],$$

where $h \in (0, T)$ *is the unique solution of* $a = c_1(h)$.

PROOF. From Hölder's inequality $\int |f|^\tau |g|^{1-\tau}\,dP \geqq \int |f|^\tau |g|^{1-\tau}\,dP_A \geqq [\int |f|\,dP_A]^\tau \cdot [\int |g|\,dP_A]^{1-\tau}$, $\tau > 1$, $P_A(B) = P(A \cap B)$ for $A, B \in \mathscr{A}$, it follows for $f = e^{tW_n}/m_n(t)$, $t \in (0, T)$, $g \equiv 1$, $A = \{W_n > na_n\}$ that

$$P(W_n > na_n) \cdot [P_t(W_n > na_n)]^{\tau/(1-\tau)} \geqq [m_n(t\tau)/m_n^{\tau}(t)]^{1/(1-\tau)} \tag{1}$$

holds for $\tau > 1$ and $t\tau \in (0, T)$.

Here P_t is defined to be the conjugate distribution of P, i.e. $P_t(B) = \int_B f\,dP$ for $B \in \mathscr{A}$ and $t \in (0, T)$. Fixing t for $t\tau \in (0, T)$ and expanding $[1/(1-\tau)] \ln m_n(t + (\tau-1)t)$ in a Taylor series about the point t and using (i), (ii) and (iv) one gets the existence of a real number $\sigma = \sigma(t)$, such that

$$[m_n(t\tau)/m_n^{\tau}(t)]^{1/(1-\tau)} = \exp n[(c_0(t) - tc_1(t)) + o(1) + (\tau-1)O(1)] \tag{2}$$

holds for $\tau > 1$ and $\tau - 1 < \sigma(t)$. From the trivial inequality $P_t(W_n \leqq na_n) \leqq P_t[e^{tW_n}/m_n(t) \leqq (e^{(t-h)na_n + hW_n})/m_n(t)] \leqq [m_n(h)/m_n(t)] \cdot e^{(t-h)na_n}$ for $t > h$ it follows from (i), (ii) and (iv) using Taylor series expansion of $\Psi_n(t)$ about the point h that

$$P_t(W_n \leqq na_n) \leqq \exp n[[(t-h)^2/2] \cdot (-c_2(h) + [(t-h)/3] \cdot C) + o(1)]$$

holds for $t > h$ in a neighborhood $U(h)$ of h. Here $C = C(h)$ is a constant (i.e. independent of n and $t \in U(h)$) satisfying $n^{-1}\,|\Psi_n^{(3)}(t)| \leqq C$ for $t \in U(h)$. Therefore

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