The Annals of Mathematical Statistics
1971, Vol. 42, No. 4, 1262–1272

THE TOPOLOGY OF DISTINGUISHABILITY[1]

BY LLOYD FISHER AND JOHN W. VAN NESS
University of Washington

1. Introduction. Hoeffding and Wolfowitz [6] consider the following problem. Independent identically distributed observations are sequentially taken on a random vector X with distribution function F. All that is known is that F belongs to a given family $\mathscr{J}$. It is desired to eventually decide either $F \in \mathscr{G}$ or $F \in \mathscr{H}$, where $\mathscr{G}$ and $\mathscr{H}$ are disjoint subsets of $\mathscr{J}$, in such a way that if $f \in \mathscr{G} \cup \mathscr{H}$ the probability of error is less than any preassigned number greater than zero.

Freedman [4] studies a modification of this problem which supposes that F is known only to be a member of a countable family $\mathscr{J}$ and it is desired to eventually decide with prescribed accuracy which member of $\mathscr{J}$ is F. This same problem is also considered in [2] and [3].

In this paper the framework is such that one would like to distinguish between a countable number of families of probability distributions, thus including both of the approaches mentioned above. Section 2 shows that under certain restrictions there is an appropriate topology associated with this question. The results of Section 2, although posed in a more general framework than Freedman's work, are essentially an easy extension of his paper. LeCam and Schwartz [8] consider estimation problems which are relevant to distinguishability by appropriate modifications.

Section 3 considers the problem of the metrizability of the topology considered. Necessary and sufficient conditions (that are somewhat difficult to apply) are given. An important example is given to show that often the topology may be nonmetrizable. The remainders of Section 3 and Section 4 give cases where the topology is metrizable and describe appropriate metrics.

2. The topology of distinguishability. Let Ω be a set and $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{A}_3 \subset \cdots$ be a non-decreasing sequence of σ-fields on Ω. Let $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n \subseteq \mathscr{B}$ where $\mathscr{B}$ is a σ-field. Take $\mathscr{M}$ to be a set of probability measures on $(\Omega, \mathscr{B})$.

Let $\mathscr{C} = \{C_\alpha, \alpha \in A\}$ be a family of disjoint subsets of $\mathscr{M}$. A decision rule for $\mathscr{C}$ written Φ, is a collection of functions ϕ_n^{α}, $\alpha \in A$, $n = 1, 2, \cdots$ on Ω with the following properties:

(a) each $\phi_n^\alpha(\cdot)$ is a real measurable function on $(\Omega, \mathscr{A}_n)$,
(b) $\phi_n^\alpha \geqq 0$,
(c) $\phi_{n+1}^\alpha \geqq \phi_n^\alpha$, and
(d) $\sum_{\alpha \in A} \phi_n^\alpha \leqq 1, n = 1, 2, \cdots$.

Received February 24, 1969; revised December 4, 1970.
[1] Research supported by NSF Grants GP-7519 and GP-13163 at the University of Washington.