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# A LIMIT THEOREM FOR A BRANCHING PROCESS WITH STATE-DEPENDENT IMMIGRATION

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Let $\{p_j\}$ and $\{a_j\}$ be probabilities defined on the non-negative integers. Consider a Markov chain $\{Z_n; n = 0, 1, \cdots\}$ defined on the non-negative integers, with stationary transition probabilities given by

$$p_{ij} = P(Z_{n+1} = j \mid Z_n = i) = p_j^{(i*)}, \qquad i \geqq 1 \tag{1}$$
$$= a_j, \qquad i = 0$$

where $p_j^{(i*)}$ is the $j$th term in the $i$th fold convolution of the sequence $\{p_j\}$. $\{Z_n\}$ represents the sizes of successive generations in a Galton–Watson process, modified to allow an immigration of particles whenever the zero state is reached. After entering the system in accordance with the probabilities $\{a_j\}$, immigrating particles reproduce with offspring law $\{p_j\}$, independently of each other and of particles already present. $Z_0$ is considered to be positive and non-random.

The case in which the offspring distribution has mean one and finite variance is of particular interest because the limit behavior of $\{Z_n\}$ is unlike that usually observed in branching processes (with or without immigration), in that a non-linear normalization is required in order to get a proper non-zero limit distribution. If the offspring mean is greater than or less than one, the appropriate normalization for $Z_n$ is the same as for the process in which the immigration is not state dependent (Pakes, Theorems 3 and 10).

From now on it will be assumed that:

1. $\sum jp_j = 1$;
2. $\sum j^2 p_j < \infty$;
3. $\sum ja_j \equiv a < \infty$;
4. $p_j < 1$ for all $j$;
5. $a_0 < 1$.

The last two assumptions are made to avoid trivial cases. An example will be given to show that $Z_n$ can have an entirely different kind of limit behavior if $a$ is allowed to be infinite.

Set $f(s) = \Sigma p_j s^j$, $h(s) = \Sigma a_j s^j$, $|s| \leqq 1$, and let $f_n(s)$ denote the $n$th functional iterate of $f(s)$ (i.e., $f_0(s) = s$ and $f_{n+1}(s) = f(f_n(s))$ for $n \geqq 0$). A simple computation

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