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A REPRESENTATION OF INDEPENDENT INCREMENT PROCESSES WITHOUT GAUSSIAN COMPONENTS[1]

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1. Introduction and summary. It is the purpose of this paper to describe a simple way of representing processes with independent increments having no Gaussian components and no fixed points of discontinuity. As is well known, the only random part of such processes are the jump discontinuities occurring at random points with random heights. The representation appearing in this paper describes the joint distribution of the ordered heights of the jumps and of the points at which these jumps occur. In fact, such a process in represented as a countable sum of functions each with one random point of discontinuity at a random height (Formula (7)). There is an analogy to the way that Wiener [8] described the Brownian motion process W_t on the interval $[0, \pi]$ as a countable sum,

$$W_t = tY_0 + 2^{\frac{1}{2}} \sum_{m=1}^{\infty} Y_m \frac{\sin mt}{m}, \tag{1}$$

where $Y_0, Y_1, \cdots$ are independent normal random variables with zero means and unit variances. In the same way that certain almost sure properties of the sample paths of the Brownian motion process can be read from (1) (see, for example, Itô and McKean [4] page 21), so also may certain almost sure properties of the general process with independent increments be read from (7).

We use the notation $\mathscr{P}(\lambda)$ to represent the Poisson distribution with parameter λ, $\mathscr{G}(\alpha, \beta)$ to represent the gamma distribution with shape parameter α and scale parameter β, $\mathscr{U}(\alpha, \beta)$ to represent the uniform distribution on the interval (α, β), and $\mathscr{N}(\mu, \sigma)$ to represent the normal distribution with mean μ and variance σ^2. (See [2] Section 3.1 for this notation.) $I_S(x)$ denotes the indicator function of the set S: one if $x \in S$, and zero if $x \notin S$. Expectations with subscripts always represent conditional expectations given the subscripted variables. $\mathbb{R}$ represents the real line and $\mathbb{R}^m$ Euclidean m-dimensional space.

Let X_t denote a process with independent increments and no fixed points of discontinuity. For the purposes of this paper, we restrict the domain of t to be the interval $[0, 1]$, and assume that $X_0 \equiv 0$. As is well known the increments of such processes have infinitely divisible distributions. Let $\psi_t(u)$ denote the logarithm of the characteristic function of X_t. The Lévy representation [5] of ψ_t may be written as

$$\psi_t(u) = ium(t) - \lambda(t)u^2 + \int_{-\infty}^{0} \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) dM_t(z)$$
$$+ \int_0^{\infty} \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) dN_t(z)$$

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