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Rejoinder

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We are grateful to the discussants for providing very valuable and insightful comments. Next, we present our views on some of the comments of the discussants and provide further discussion.

We thank Bevilacqua, Hering and Porcu (hereafter, BHP) for bringing attention to the fundamental problem of comparing multivariate models. Until now, almost all comparisons between models have been relegated to empirical performance on specific datasets, whether it be performance on cokriging or particular scoring rules. BHP introduce two theoretical approaches to comparing the flexibility of multivariate frameworks: (A) assessing the size of allowable co-located cross-correlation between processes, and (B) a measure of difference in allowed spatial (cross)-correlation at differing distances.

Regarding (A), BHP claim the bivariate Matérn is less flexible than the LMC in that there are nontrivial restrictions on the cross-correlation coefficient for the bivariate Matérn that are not present for the LMC. We emphasize, however, that the bivariate Matérn restrictions are a *characterizing* feature of the covariance class—no LMC construction can allow for marginal and cross exact Matérn behavior while allowing for unrestricted choice of co-located cross-correlation. Rather, it is a physical restriction on the covariance class, not a flexibility restriction.

Most spatial modelers include a nugget effect in the statistical model, $Y_i(\mathbf{s}) = Z_i(\mathbf{s}) + \varepsilon_i(\mathbf{s})$, where $Z_i(\mathbf{s})$ is endowed with a multivariate model, and $\varepsilon_i(\mathbf{s})$ is a white noise process that is uncorrelated with $Z_i(\mathbf{s})$. If $\varepsilon_i(\mathbf{s})$ is nontrivial with variance τ_i^2, then the restrictions on the cross-correlation coefficient can be relaxed, the amount depending on the magnitude of the nugget effect and sample size. To see this, let $p = 2$ and write the covariance matrix for two unit variance processes at n locations $\{Z_1(\mathbf{s}_1), \ldots, Z_1(\mathbf{s}_n), Z_2(\mathbf{s}_1), \ldots, Z_2(\mathbf{s}_n)\}^{\mathrm{T}}$ as $B \odot \Sigma$, where $\Sigma = \{C_{ij}(\mathbf{s}_k, \mathbf{s}_\ell)\}_{i,j=1;k,\ell=1}^{2;n}$ and $B = (B_{ij})_{i,j=1}^2$ consists of four $n \times n$ block matrices. For simplicity, assume $\tau_1 = \tau_2$, so that $B_{12} = B_{21}$ are matrices populated by a constant ρ_0 and $B_{11} = B_{22}$ are matrices of ones with diagonal $1 + \tau^2$. Note that the case $\rho_0 = 1$ results in $B \odot \Sigma$ having the specified multivariate dependence; if $\rho_0 > 1$, then the two processes can have larger cross-correlation than allowed by the specified model. The cases where $\rho_0 > 1$ are valid when $B_{12} = B_{11}^{1/2} K B_{22}^{1/2}$, where K is a contraction matrix (i.e., a matrix whose singular values are bounded by unity); this follows from Proposition 1 of Kleiber and Genton (2013). This is one feasible way to relax the restrictions that are suggested by BHP's (A) criterion. We view BHP's (B) as an alternative interesting route to comparing models, although it is still unclear what improvements a modeler would expect to gain for various magnitudes of the (B) criterion.

Cressie et al. focus on three main aspects: the importance of modeling the nugget effect (which yields additional potential difficulties in the multivariate context), the pseudo cross-variogram and alternative approaches to building multivariate structures.

We focused our efforts on reviewing multivariate covariance functions, not multivariate modeling, a byproduct of which is that we left little discussion to the issue of modeling the nugget effect. For instance, the underlying latent smooth process $\mathbf{W}$ of Cressie et al. [(2015), equation (4)] still requires specification of the multivariate structure, regardless of whether a nugget effect will or will not ultimately be included. Nonetheless, these authors bring up an important point in that, especially for multivariate processes, some variables may be measured by the same instrument, in which case it may be expected that measurement errors are correlated across variables at individual locations. Disentangling microscale variability of the process from measurement error is indeed a difficult prospect; Sang, Jun and Huang (2011) used a full-scale approximation for multivariate processes that explicitly breaks up large scale, small scale and measurement error variability.

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