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Nonparametric Inference for Max-Stable Dependence

Johan Segers

The choice for parametric techniques in the discussion article is motivated by the claim that for multivariate extreme-value distributions, "owing to the curse of dimensionality, nonparametric estimation has essentially been confined to the bivariate case" (Section 2.3). Thanks to recent developments, this is no longer true if data take the form of multivariate maxima, as is the case in the article. A wide range of nonparametric, rank-based estimators and tests are nowadays available for extreme-value copulas. Since max-stable processes have extreme-value copulas, these methods are applicable for inference on max-stable processes too. The aim of this note is to make the link between extreme-value copulas and max-stable processes explicit and to review the existing nonparametric inference methods.

1. EXTREME-VALUE COPULAS

Let the random variables $Y_1, \ldots, Y_D$ represent the maxima in a given year of a spatial process (e.g., rainfall) that is observed at a finite number of sites, $x_1, \ldots, x_D$, in a region $\mathcal{X}$ in space $\mathbb{R}^p$ (typically, $p = 2$). Let $F_1, \ldots, F_D$ be the marginal cumulative distribution functions, assumed to be continuous. In the article, these are assumed to be univariate generalized extreme-value distributions, an assumption that will not be needed here.

The random variables $U_d = F_d(Y_d)$ are uniformly distributed on the interval $(0, 1)$ and the joint cumulative distribution function C of the vector $U_1, \ldots, U_D$ is the copula of the random vector $Y_1, \ldots, Y_D$:

$$C(u_1, \ldots, u_D) = \Pr(U_1 \le u_1, \ldots, U_D \le u_D) \tag{1}$$

for $0 \le u_d \le 1$. The requirement that the random vector $Y_1, \ldots, Y_D$ is max-stable entails

$$C^m(u_1^{1/m}, \ldots, u_D^{1/m}) = C(u_1, \ldots, u_D) \tag{2}$$

for all $m > 0$. In [18], it was shown that (2) holds if, and only if,

$$C(u_1, \ldots, u_D) = \exp\{-rA(v_1, \ldots, v_D)\}, \tag{3}$$

where $r = -\sum_{d=1}^{D} \log u_d$ and $v_d = -r^{-1} \log u_d$. The domain of the Pickands dependence function A is the unit simplex, $\mathcal{S}_D = \{v \in [0, 1]^D : \sum_d v_d = 1\}$. A necessary and sufficient condition for a function A on $\mathcal{S}_D$ to be a Pickands dependence function is that

$$\begin{aligned} &A(v_1, \ldots, v_D) \\ &\quad = \int_{\mathcal{S}_D} \max(v_1 s_1, \ldots, v_D s_D)\, \mathrm{d}M(s_1, \ldots, s_D) \end{aligned} \tag{4}$$

for a Borel measure M on $\mathcal{S}_D$ verifying the constraints $\int_{\mathcal{S}_D} s_d \,\mathrm{d}M(s_1, \ldots, s_D) = 1$ for all $d \in \{1, \ldots, D\}$. In particular, A is convex and $\max(v_1, \ldots, v_D) \le A(v_1, \ldots, v_D) \le v_1 + \cdots + v_D$. In dimension $D = 2$, these two properties completely characterize Pickands dependence functions (but not if $D \ge 3$).

2. MAX-STABLE MODELS

The representation in (3)–(4) is valid for general max-stable copulas and therefore also holds for the finite-dimensional distributions of the max-stable processes considered in Section 6 in the article. The purpose of this section is to make this relation explicit.

Consider the simple max-stable process

$$Z(x) = \max_{j \ge 1}[S_j \max\{0, W_j(x)\}], \quad x \in \mathbb{R}^p, \tag{5}$$

where $\{S_j\}_{j=1}^{\infty}$ are the points of a Poisson process on $\mathbb{R}_+$ with rate $s^{-2}\,\mathrm{d}s$ and where $W_1, W_2, \ldots$ are iid replicates of a stationary stochastic process W on $\mathbb{R}^p$, independent of the previous Poisson process, and such that $\mathrm{E}[W^+(x)] = 1$, where we write $W^+(x) = \max\{0, W(x)\}$. Particular cases of this model include the so-called Smith model [24], the Schlather model [22] and the Brown–Resnick model [12].

The stationary, marginal distribution of $Z(x)$ in (5) is unit-Fréchet and the joint distribution function of the

Johan Segers is Professor, Institut de statistique, biostatistique et sciences actuarielles, Université catholique de Louvain, Voie du Roman Pays 20, B-1348 Louvain-la-Neuve, Belgium (e-mail: johan.segers@uclouvain.be).