in the linear $\mathbf{x}_i - \bar{\mathbf{x}}$ term relating A to $\mathbf{y}_i - \bar{\mathbf{y}}$ (1). With this model the δ should be compared directly, with triangle dimension entering only in estimating the precision.

2. For large strains $d\mathbf{Q}$ is not invariant (to within a rotation) to choice of base edge (10).

3. One difficulty with the approach via triangles is that measurement error is propagated through the choice of a common baseline (in a quadrilateral say) whereas variance considerations suggest averaging. Only for each triangle of landmarks is the translation of a single landmark always equal to the translation of the landmark centroid, $\mathbf{y}_i - A\bar{\mathbf{x}}_1 = \bar{\mathbf{y}} - A\bar{\mathbf{x}}$.

The modelling inconsistencies in using triangles have already been considered. However, this approach does define a basis direction comparable across individuals and does extend the homology between individuals from landmarks to the whole form (using pseudo-landmarks). As mentioned previously, modelling of interindividual variation requires at least the definition of a basis direction, and some tacit notion of extended homology also. These are the principal conceptual impediments to development of a theory of morphometrics along the lines of this discussion. They are no novelty to morphometrics, and it is substantially to Bookstein's credit that he has dealt with them.

As a final remark, the next generalization is to longitudinal data, for which the positions of a set of landmarks, possibly evolving in time, are recorded at several time points. The deformation tensor field is varying in space and time. Technically, many of the issues are the same, as in fact approaches for the analysis of finite deformation have been borrowed from the analysis of longitudinal data itself.

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Rejoinder

Fred L. Bookstein

Five able discussants have persuaded me that my essay, however long already, spent too little space reviewing themes other than its own. Each discussion points out connections between morphometrics and diverse topics both within biometrics and without.

Kendall's shape space Σ_2^3. David Kendall surmises, correctly, that I had not previously encountered his work. Indeed we have approached nearly the same problem from two very different directions. Although permutations and reflections of landmark configurations are prohibited on biological grounds, the algebra of my shape space is still that of a tangent space at the point of his Σ_2^3 corresponding to the mean shape. In the large, the plane of shape coordinates Q represents all of his shape space, except for one point $Z_1 = Z_2$, six times over.

The tensors supply a canonical geometric description of directions in any tangent plane of this space. Also, they lead to a metric geometry throughout the space, with infinitesimal element of distance equal to $|dQ|/\text{Im } Q$, the difference of the log principal strains. The geodesic arcs of this geometry are curves corresponding to triangular shapes whose transformations from a fixed starting triangle have the same principal axes—the shapes that can be reached by fractional powers of the same affine transformation. The geodesics, then, must be the circles involved in the construction of the principal axes (Figure 11), the circles orthogonal to the real axis. In this metric construction for shape space we recognize one of the classic models of hyperbolic geometry, the Poincaré half-plane (cf. Coxeter, 1965, Section 14.8).

Such a space has negative Riemannian curvature, whereas Kendall's shape space, under the metric inherited from Euclidean distance, has positive curvature. This and other interesting differences between Kendall's geometry and mine derive from the differences between their fields of principal application. My morphometric shape space has a line of metric singularities all down the real axis. Points (triangles) not on this axis may not be transformed into points upon it or across it by any proper affine transformation. That axis, representing triangles of zero area, is the Absolute of the hyperbolic geometry, the locus infinitely far away. Its exclusion expresses the restriction of the deformation model to transformations of