Comment

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Professor Reid provides a stimulating review of the theory and application of saddlepoint methods in parametric statistical analysis. As is indicated in Sections 6.4 and 6.6, similar approximations can be applied to certain nonparametric statistical calculations. Robinson (1982) applies the saddlepoint technique to obtain approximations to permutation distributions, and more recently Davison and Hinkley (1988) have applied saddlepoint approximations to several bootstrap and randomization problems. Great numerical accuracy is evident in most of these applications. The corresponding theoretical development, which requires some delicacy, is contained in Wang's Ph.D. dissertation for statistics which are sums of random variables. We should like to summarize and illustrate some of the results for a simple bootstrap problem here.

Let $(X_1, \ldots, X_n)$ be independently sampled from the continuous distribution function F whose mean is $\mu = E(X_1)$. Suppose that we wish to calculate the cumulative distribution function (CDF) G of the estimation error $D = \bar{X} - \mu$, where $\bar{X} = n^{-1} \sum X_i$. If F is known, and if the cumulant generating function $K(t) = \log\{\int_{-\infty}^{\infty} e^{t(x-\mu)}\, dF(x)\}$ exists in a neighborhood of $t = 0$ and is calculable, then a saddlepoint formula will give a very accurate approximation to G (see Section 6.3).

But suppose that F is completely unknown. The bootstrap approach (Efron and Tibshirani, 1986) is to calculate G with the empirical CDF $\tilde{F}$ in place of F. That is, one estimates G by $\tilde{G}$, the CDF of $\bar{X}^* - \bar{x}$ when $\bar{X}^*$ is the average of $(X_1^*, \ldots, X_n^*)$ which are sampled randomly with replacement from the fixed, observed set $(x_1, \ldots, x_n)$. A standard implementation of the bootstrap would approximate $\tilde{G}$ by Monte Carlo methods, e.g., by direct simulation of hundreds of samples $(X_1^*, \ldots, X_n^*)$ and calculation of empirical cumulative frequencies for $\bar{X} - \bar{x}$. Saddlepoint methods offer an alternative, efficient approach to approximation to $\tilde{G}$.

In principle some care is needed here because $\tilde{F}$, and hence $\tilde{G}$, are discrete, and slightly different saddlepoint formulas apply in discrete cases. Suppose that the x_i's are given to m decimal places, so that $\bar{X}^*$ is a multiple of $n^{-1}10^{-m}$. Define

$$\text{(A1)}\quad \begin{aligned}\tilde{K}(t) &= \log\left\{\int_{-\infty}^{\infty} e^{t(x-\bar{x})}\, d\tilde{F}(x)\right\}\\ &= \log\left[n^{-1} \sum \exp\{t(x_i - \bar{x})\}\right].\end{aligned}$$

Then the saddlepoint approximation to $\tilde{G}(d) = \Pr(\bar{X}^* - \bar{x} \le d \mid \tilde{F})$ when d is a multiple of $n^{-1}10^{-m}$ is, corresponding to Reid's equation (28),

$$\text{(A2)}\quad \tilde{G}_s(d) = \begin{cases}\Phi(w) - \phi(w)\,[10^{-m}\{1 - e^{10^{-m}T}\}^{-1} \\ \quad \cdot \{n\tilde{K}''(T)\}^{-1/2} - w^{-1}], & d_1 \neq 0,\\ 1/2 + 1/6(2\pi n)^{-1/2}\{\tilde{K}''(0)\}^{-3/2}\tilde{K}'''(0) \\ \quad - 1/2\,10^{-m}\{2\pi n\tilde{K}''(0)\}^{-1/2}, & d_1 = 0,\end{cases}$$

where $d_1 = d + n^{-1}10^{-m}$, $\tilde{K}'(T) = d_1$ and

$$w = [2n\{Td_1 - \tilde{K}(d_1)\}]^{1/2}\operatorname{sgn}(T).$$

Wang has proved that

$$\text{(A3)}\quad \tilde{G}(d) = \tilde{G}_s(d)\{1 + O_p(n^{-1})\},$$

but that the relative error is *not* strictly uniform in the tails for fixed n. In this latter sense the saddlepoint approximation is not as strong as usual, although in practice this seems unimportant.

Recall that $\tilde{G}$ is itself intended to be an approximation, to the continuous CDF G. For this purpose it may be sensible to modify (A2) with a continuity correction, i.e., to approximate G by

$$\text{(A4)}\quad \tilde{G}_1(d) = \tilde{G}_s(d - 1/2\,n^{-1}10^{-m}).$$

Note that $\tilde{G}_s$ is continuous.

A somewhat more casual approach is to ignore the discreteness, and to apply Reid's (28) with $\tilde{K}$ as in (A1) replacing K. We denote the result by $\tilde{G}_2$. In fact, as the following numerical example shows, there will often be negligible differences among $\tilde{G}_s$, $\tilde{G}_1$ and $\tilde{G}_2$.

The numerical example involves the sample of $n = 10$ numbers, with $m = 1$,

9.6 10.4 13.0 15.0 16.6 17.2 17.3 21.8 24.0 33.8.

Approximate percentage points for $\bar{X}^* - \bar{x}$ have been calculated using $\tilde{G}_s$, $\tilde{G}_1$ and $\tilde{G}_2$. Some of the results are

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