Comment

Evarist Giné and Joel Zinn

Since Dudley's influential paper of 1978 the theory of empirical processes has undergone a vigorous development. David Pollard and his collaborators, among others, have applied some of these developments in asymptotic statistics. However, probably due to the technical character of this theory, applications are slow in coming. The present article will certainly help to make the subject better known to potential users.

We have no criticism to offer on this interesting paper. Instead, we take it as a basis for a digression both on points of view and aspects of empirical process theory that we have found useful in our work.

In the present article, Pollard describes how to obtain maximal inequalities for Gaussian and related processes using the "chaining method" associated to metric entropy. It is important to highlight this subject as Pollard has done, because, directly or indirectly, it is at the core of most of the progress made on empirical processes since 1978. Closely connected to this subject is Talagrand's (1987a) landmark work characterizing sample boundedness and continuity of Gaussian processes by means of properties of their covariances. These properties are the so called majorizing measure conditions which, like metric entropy, are conditions on the size of the index set for the Gaussian pseudodistance. Actually these are the minimal conditions under which a chaining proof quite similar to the one here can still be carried out (see, e.g., Remark 2.6 in Andersen, Giné, Ossiander and Zinn, 1988). Rhee and Talagrand (1988) show how this more refined chaining method can be of practical interest. They obtain a precise maximal inequality for an empirical process in a concrete situation with implications in bin packing by constructing the appropriate majorizing measure. More applications of majorizing measures to empirical processes, in connection with bracketing, can be found in Andersen, Giné, Ossiander and Zinn (1988).

Given the wealth of results available for Gaussian processes, notably deviation and concentration inequalities, integrability, comparison theorems, and, of course, the already mentioned maximal inequalities, direct and converse (for references see, e.g., Pisier, 1986; Giné and Zinn, 1986), it is sometimes effective to hypothesize Gaussian properties for $\mathscr{F}$ either instead of, or in conjunction with, more analytical conditions such as entropy. As a very naive instance, here is a Gaussian definition of manageable class weaker than Definition 4.3 and which does essentially the same job. Let $\mathscr{F}$ be a class of functions with envelope F. For $Q = \sum \alpha_i \delta_{s_i} \in \mathscr{P}_f(S)$, the set of probability measures on S with finite support, define the Gaussian process $W_Q(f) = \sum \alpha_i^{1/2} g_i f(s_i)/(QF^2)^{1/2}$, where g_i are i.i.d. $N(0, 1)$, and let $W_Q(f, g) = [E(W_Q(f) - W_Q(g))^2]^{1/2}$. Then say that $\mathscr{F}$ is manageable if

(i) $\sup_{Q\in\mathscr{P}_f(S)} E\| W_Q\|_{\mathscr{F}} < \infty$ and
(ii) $\lim_{\delta\to 0} \sup_{Q\in\mathscr{P}_f(S)} E\| W_Q \|_{\mathscr{F}'(\delta, W_Q)} = 0$

where $\mathscr{F}'(\delta, W_Q) = \{f - g: f, g \in \mathscr{F}, W_Q(f, g) \le \delta\}$. Stability properties and results similar to Theorems 4.5 and 4.7 still hold for such classes. For instance, a proof of (a weaker form of) Corollary 4.6 goes as follows: using property (i) together with symmetrization and Jensen's inequality as in the text, we have

$$E\|\nu_n\|_{\mathscr{F}} \le 2E \left\| n^{-1/2} \sum_{i=1}^n \sigma_i f(X_i) \right\|_{\mathscr{F}}$$
$$\le \sqrt{2\pi}\, E \left\| n^{-1/2} \sum_{i=1}^n g_i f(X_i) \right\|_{\mathscr{F}}$$
$$= \sqrt{2\pi}\, E[(P_n F^2)^{1/2} E_g \| W_{P_n} \|_{\mathscr{F}}] \le C(PF^2)^{1/2}.$$

The proof of Theorem 4.7 would use (ii) and a comparison theorem of Fernique (1985). If $\mathscr{F} = \{f_n: \|f_n\|_\infty = o(1/(\log n)^{1/2})\}$ then $\mathscr{F}$ is manageable in this weaker sense but not necessarily in the sense of Definition 4.3. For more details on these classes of functions, see Giné and Zinn (1989).

In the applications presented by Pollard, error terms in Taylor expansions are controlled by the size of $\|\nu_n\|_{\mathscr{F}}$, the sup of the empirical process over a class of functions $\mathscr{F}$, and therefore probability inequalities for $\|\nu_n\|_{\mathscr{F}}$, i.e., maximal inequalities, yield the desired results. Other types of possible applications of empirical processes would relate to the construction of asymptotic confidence regions and tests of hypotheses based on the statistics $\|P_n - P\|_{\mathscr{F}} = n^{-1/2}\|\nu_n\|_{\mathscr{F}}$. These would require knowledge of the limiting distribution of $\|\nu_n\|_{\mathscr{F}}$ or, in general, of the limiting law of

Evarist Giné is Professor of Mathematics at Texas A & M University. He is presently on leave at The College of Staten Island, City University of New York. His address is Department of Mathematics, The College of Staten Island, 130 Stuyvesant Place, Staten Island, New York 10301. Joel Zinn is Professor, Department of Mathematics, Texas A & M University, College Station, Texas 77843-3368.