case prior opinion may be incorporated informally, subsequent to the analysis. In any case, the distribution assigned to β and θ should be regarded as part of the model.

5. The posterior distribution of w (i.e., the conditional distribution of w given y) may suggest suitable point and interval predictors. With the possible exception of the term posterior distribution, which might be used in referring to any distribution that is conditional on y, the use of Bayesian jargon should be avoided.

Comment: The Kalman Filter and BLUP

James C. Spall

1. INTRODUCTION

Professor Robinson has given a wide-ranging account of best linear unbiased prediction with an impressive array of examples and applications. In this discussion, however, I will restrict my attention to issues regarding the Kalman filter and BLUP.

For ease of discussion, let us restate the random effects model in state-space form as given in Robinson, Section 6. The unobservable random effects (state) vector, u_t, evolves according to

$$(1.1a) \qquad u_t = G_t u_{t-1} + w_t,\ u_0 = 0,\ t = 1, 2, \ldots, n,$$

where w_t is a noise term with mean 0 and covariance matrix W_t, and G_t is the state transition matrix. The second equation in the model relates the state vector to the vector of observables y_t:

$$(1.1b) \qquad y_t = F_t u_t + v_t,$$

where v_t is a noise term with mean 0 and covariance matrix V_t, and F_t is the measurement matrix. Equations (1.1a, b) can be expressed in the random effects model form of Robinson by writing

$$y = Zu + e,$$

where $y = (y_1^T, y_2^T, \ldots, y_n^T)^T$, $Z = \text{block diag}[F_1, F_2, \ldots, F_n]$, $u = (u_1^T, u_2^T, \ldots, u_n^T)^T$, and $e = (v_1^T, v_2^T, \ldots, v_n^T)^T$. The covariance matrix for u, G in the notation of Robinson, is a function of G_t and W_t, $t = 1, 2, \ldots, n$. The structure of this covariance matrix allows for recursive algorithms of the Kalman filter/smoother form to be used to form BLUP estimates for the components of u. Incidentally, a slightly confusing point in Robinson, Subsection 6.4, is that it is a Kalman smoother, not filter, that produces the BLUP estimate of u based on data y. What Robinson had in mind, I presume, is the common problem where one is interested in an estimate of u_t based only on data through time t (not through some later time); the Kalman *filter*, of course, is used for this problem. For the remainder of this discussion, I will assume that the filtering problem is the one of interest (although virtually all of the ideas would also apply in the smoothing problem).

A couple of other points are worth noting here. First, Sallas and Harville (1988) address a slightly broader problem than that considered above and by Robinson: namely the estimation of random *and* fixed effects via Kalman filter techniques. Second, as noted by Robinson, the Kalman filter is not entirely due to Kalman. The filter equations were essentially derived by others prior to Kalman, but it was Kalman who crystallized much of the thinking in the area and discovered several key relationships to certain systems-theoretic concepts (see Spall, 1988, for further discussion of this).

In the next two sections, I will discuss two problems that were given fairly light treatment in the Robinson paper, but that are important from the point of view of a practitioner. Section 2 describes some problems associated with constructing uncertainty bounds for the filter estimation error $\hat{u}_n - u_n$ when the noise terms have an unknown distribution (as in the general setting of Robinson, equation 1.1). Section 3 elaborates on the brief discussion of Robinson regarding uncertainty in the model parameters θ.

2. UNCERTAINTY BOUNDS FOR $\hat{u}_n - u_n$ IN DISTRIBUTION-FREE SETTINGS

Robinson presents the formula for the covariance matrix of the BLUP estimation error in Section 1 of his paper, and it is well known that this covariance

James C. Spall is a member of the Senior Professional Staff at The Johns Hopkins University, Applied Physics Laboratory, Laurel, Maryland 20723-6099.