Theorem 2.1), which says that if $R(x \to y) = R(y)$ and

$$\pi\left(x\colon \frac{R(x)}{\pi(x)} \le \frac{1}{m}\right) > 0$$

for all m, then $\|P(X^{(t)} = \cdot|x^{(1)}) - \pi(\cdot)\|$ tends to zero in t slower than geometrically. It is straightforward to check these conditions in the Gaussian example, and hence convergence is very slow.

With a random proposal density we can get a geometrically convergent MCMC: Let $R(x \to y) = R(y)$ be, with probability $\frac{1}{2}$, a multivariate normal $\mathscr{N}(\mu, \Sigma^{-1})$ and, with probability $\frac{1}{2}$, a multivariate normal $\mathscr{N}(-\mu, \Sigma^{-1})$. To bound the rate of convergence one can use directly the uniform minorization technique in Roberts and Polson (1994). Since

$$P(x \to y) \ge \pi(y)\exp\left[-\tfrac{1}{2}\mu^T \Sigma \mu\right],$$

it follows that

$$\|P(X^{(t)} = \cdot|x^{(1)}) - \pi(\cdot)\| < \left(1 - \exp\left(-\tfrac{1}{2}\mu^T \Sigma\mu\right)\right)^t,$$

and convergence is geometric. Hence, randomizing the proposal density helps. The mixture is somehow reminiscent of antithetic variables. We get a burn-in of order $O(\exp(\frac{1}{2}\mu^T \Sigma\mu))$, which may be quite overestimated because the uniform minorization technique is sometimes poor. Consider again, for instance, the two-dimensional Ising model with T sufficiently large. For a uniform proposal probability the best estimate of the burn-in for Metropolis, based on uniform minorization, is $O(\exp[(2/T)n])$, while one can show in this case (see Frigessi, Martinelli and Stander, 1993) that always $t^* \le O(e^{c\sqrt{n}})$ and under condition (MO) in that paper $t^* = O(n \log n)$. For the Gibbs sampler the bound is even worse.

The next simple example shows that sometimes a random proposal density does not speed up convergence w.r.t. a deterministic density. Take π to be the exponential density with parameter λ. Let $R(x \to y) = R(y)$ be also exponential with parameter $0 < \lambda' < \lambda$. Then the acceptance probability is

$$A(x \to y) = \min(1, \exp[-(\lambda - \lambda')(y - x)])$$

and the uniform minimization bound yields

$$\|P(X^{(t)} = \cdot|x^{(1)}) - \pi(\cdot)\| \le \left(1 - \frac{\lambda'}{\lambda}\right)^t.$$

As before, consider now the random proposal density (again a symmetric mixture)

$$R(x \to y) = R(y) = \tfrac{1}{2}(\lambda' \exp(-\lambda' y) + (2\lambda - \lambda')\exp[-(2\lambda - \lambda')y]).$$

Via uniform minimization we obtain

$$\|P(X^{(t)} = \cdot|x^{(1)}) - \pi(\cdot)\| \le \left(1 - \frac{\lambda'}{2\lambda}\right)^t > \left(1 - \frac{\lambda'}{\lambda}\right)^t.$$

Under a prudent policy, that is, trusting only certain bounds, here in this example randomizing can slow down convergence. Of course lack of symmetry plays a role. Summarizing, a blind use of random proposal densities may not be advantageous. Are there some guidelines for a successful application of this potentially powerful idea?

Comment

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We heartily endorse the authors' conclusion that Markov chain Monte Carlo (MCMC) "represents a fundamental breakthrough in applied Bayesian modeling." We laud the authors' effective unification of spatial, image-processing and applied Bayesian literature, with illustrative examples from each area and a substantial reference list. (As an aside, one of us pondered the significance of the fact that roughly one-fourth of the entries in this list have lead authors whose surname begins with the letter "G"!)

We begin with a few preliminary remarks. First, with regard to practical implementation, the artificial "drift" among the variables alluded to in Section 2.4.3 is well known to those who fit structured random effects models and is a manifestation of weak identification of the parameters in the joint posterior. Reparametrization and more precise hyperprior specification are common tricks to improve

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