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Consider the following simplified version of Freedman's (1963) example showing inconsistency of the posterior distribution for countable parameters.

Let θ_∞ denote the geometric distribution

$$p(i) = 3/4^{i+1}, \quad i = 0, 1, \dots$$

Let θ_k denote the truncated geometric distribution

$$p(i) = 3/4^{i+1}, \quad i = 0, 1, \dots, k,$$

$$p(i) = 0, \quad i = k + 1, \dots,$$

$$p(i) = 1/4^{k+1}, \quad i = -1.$$

Let θ_0 denote the geometric distribution

$$p(i) = 3^i/4^{i+1}, \quad i = 0, 1, \dots$$

A prior distribution gives probability Π_k to θ_k . The likelihood of X_1, X_2, \dots, X_n at θ_k is $(\frac{3}{4})^n (\frac{1}{4})^{S_n}$ for $0 < k, k \geq \max X_i$, and is $(\frac{1}{4})^n \cdot (\frac{3}{4})^{S_n}$ at $k = 0$, where $S_n = \sum X_i$.

The posterior probability of θ_0 is

$$\Pi_0 3^{S_n - n} / \left(\Pi_0 3^{S_n - n} + \sum_{k \geq \max X_i} \Pi_k \right).$$

When θ_∞ is true, $S_n = \frac{1}{3}n + O(\sqrt{n})$, $\max X_i \sim \log_4 n$. Let $Q_k = \sum_{j=k} \Pi_j$. Choose the prior Π so that $\Pi_k > 0$ all k , $3^n Q(\log_4 n) \rightarrow 0$ as $n \rightarrow \infty$. Thus θ_∞ is true, and every neighborhood of θ_∞ has positive probability in the weak-star topology, yet the posterior probability of θ_0 converges to 1.

From this and other arguments the authors conclude that posteriors are usually inconsistent when the parameters are countably dimensional. But note that the parameter space here is just the integers $k = 0, 1, \dots, \infty$ (and in Freedman's example it is a closed interval). The counterexample applies equally well in finite dimensional cases if one induces a topology on the parameters by the weak-star topology on distributions!

The weak-star topology gives little weight to the tails, but small differences in the tails can have very large relative effects on the likelihood, and therefore on the posterior probabilities. I wonder if it wouldn't be possible to escape from the inconsistency by using a likelihood-friendly topology such as one based on the distance

$$\rho(P, Q) = \sum_{i=0}^{\infty} (P_i - Q_i) \ln \frac{P_i}{Q_i} ?$$

In this topology, the θ_i of the example are isolated points, and θ_∞ has zero probability, so there is no surprise in finding the posterior inconsistent.

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