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NOTE

ADDENDUM TO "PROPERTIES OF HERMITE SERIES ESTIMATION OF PROBABILITY DENSITY"

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The results presented in this work (*Ann. Statist.* **5** 1258–1264) deal only with rates of mean square convergence and integrated mean square convergence of the estimator based on Hermite series. It is possible to obtain similar results for almost sure convergence. Indeed this has already been done by D. Bosq for the same estimator. He showed that for densities satisfying conditions similar to those hypothesized here, the Hermite series estimators and their derivatives are uniformly convergent almost surely. His results can be improved slightly, as were those of Schwartz, by using better bounds on the Hermite functions. He also showed that the derivatives of the estimator are integrated mean square convergent, but did not calculate rates in either case.

The rates of almost sure convergence can be obtained from the mean square rates by using a result of Kiefer, that the empiric distribution function $\hat{F}_n$ satisfies

$$\sup_x|\hat{F}_n(x) - F(x)| = 0\left(\left(\frac{\log\log n}{n}\right)^{\frac{1}{2}}\right) \text{ a.s.}$$

This may be combined with Theorem 3, for instance, to obtain for densities such that $f^{(r)} \in L_2 \cap C_0$, $x \in K$, a compact set,

$$\sup_x|\hat{f}_n^{(p)}(x) - f^{(p)}(x)| \leqslant \sup_x|\hat{f}_n^{(p)}(x) - E\hat{f}_n^{(p)}(x)| + \sup_x|E\hat{f}_n^{(p)}(x) - f^{(p)}(x)|$$
$$\leqslant \sup_x|\textstyle\int\sum_{k=0}^q h_k^{(p)}(x)h_k(t)(d\hat{F}_n(t) - dF(t))| + 0(n^{(p-r+1)/2r}),$$

since the bias is less than the square root of the mean square error. The remaining integral may be attacked by using integration by parts to obtain

$$\sup_x|\hat{f}_n^{(p)}(x) - f^{(p)}(x)| \leqslant \sup_{x,t}|\hat{F}_n(t) - F(t)|\textstyle\sum_{k=0}^q\int|h_k^{(p)}(x)h_k'(t)|\,ds$$
$$+0(n^{(p-r+1)/2r}) \leqslant 0\left(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}\right)[q(n)]^{\frac{p}{2}+\frac{7}{4}} + 0(n^{(p-r+1)/2r}) \text{ a.s.}$$

If $q(n) = [n^{1/r}]$, as it was in the theorem, the rate of almost sure convergence is

$$0\left(n^{-\frac{1}{2}+(2p+7)/4r}(\log\log n)^{\frac{1}{2}}\right)$$

for $r = 4, 5, \ldots,$ $p = 0, 1, \ldots, r - 4$. On the other hand if $q(n) = [n^{1/(r+3)}]$ the

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