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A. M. KAGAN, YU. V. LINNIK AND C. RADHAKRISHNA RAO, *Characterization Problems in Mathematical Statistics* (translated from the Russian by B. Ramachandran). John Wiley and Sons, New York, 1973, xii+499 pages, $26.25.

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Introduction. The study of characterizations of probability distributions has had a long history. Indeed, Gauss (1809) proved (under certain restrictions) that the maximum likelihood estimator of the location parameter of a distribution is the sample mean if and only if the distribution is normal.

In general, a characterization problem takes the following form: Suppose that for a random vector X, there is a family $\mathscr{F}$ of distributions such that $\mathscr{L}(X) \in \mathscr{F}$ implies that X has a certain property $\mathscr{P}$. The characterization problem is the converse, namely, to show that if the random vector X exhibits property $\mathscr{P}$, then $\mathscr{L}(X) \in \mathscr{F}$.

There are two ingredients in a characterization problem: the family of distributions $\mathscr{F}$, and the property $\mathscr{P}$. The list of possible families includes the normal, exponential, gamma, beta, geometric, Poisson, Cauchy, and Wishart distributions, but the normal distribution receives most of the attention. The various possible properties cannot be described very succinctly, but the following brief topic headings provide some loose indications: identical distributions of specified functions of X, independence of specified functions of X, convolutions of the distribution of X, functions of X having specified regression on other functions of X, functions of X being maximum likelihood estimators (or admissible estimators) of an unknown parameter of the distribution of X.

Although Pólya proved an elegant result in 1923 characterizing the normal distribution by the identical distribution of two linear statistics, characterization problems did not begin to attract serious attention until 1935 when Paul Lévy conjectured that if $X + Y$ is normal,[1] then X and Y are normally distributed, that characterization problems began to attract some interest. Cramér (1936) proved Lévy's conjecture, while Raikov (1937) proved a similar result for the Poisson distribution. Marcinkiewicz (1939) proved a result related to that of Pólya mentioned above, and Kac (1939) and Bernstein (1941) proved that $X + Y$ independent of $X - Y$ implies that X and Y are normal. There was a modest amount of activity in the 1940's, followed by a rapid growth of activity in the 1950's.

There have been few expository works covering this material; of import are the review paper by Lukacs (1956), and the monograph by Lukacs and Laha

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[1] In the remainder of this review it will be tacitly assumed that all random variables are independently distributed.