DISCUSSION ON PROFESSOR ORNSTEIN'S PAPER[1]

Professor Paul C. Shields (*Stanford University*). Professor Ornstein's paper describes some of the properties of Bernoulli processes. In particular he shows that the B-processes are precisely the closure of the mixing n-step Markov processes in the $\bar{d}$-metric, and that entropy is a complete invariant for B-processes. In this and elsewhere it has been shown that factors and roots of Bernoulli shifts are Bernoulli and that Bernoulli shifts are embeddable in flows.

If we take larger classes of automorphisms much pathology can occur (K-automorphisms may not have roots, the entropy 0 factor may not be a direct factor). We propose here a definition of regularity which may include all that is of physical interest and excludes much of this pathology. We say that a class **R** of automorphisms is ***regular*** if it satisfies

(R1) **R** is closed in the $\bar{d}$-metric.

(R2) Each transformation in **R** is embeddable in a flow.

(R3) If $T \in$ **R**, any factor of T is in **R**.

(R4) Any transformation in **R** of positive but not completely positive entropy is the direct product of an entropy zero factor and a K-automorphism.

The class of Bernoulli shifts is a regular class. The class of rotations of the circle is also a regular class, for one can easily show that any two distinct rotations are at least $\frac{1}{2}$ apart in the $\bar{d}$-metric. (For this class (R2) and (R4) are trivial, while Adler [1] has established (R3)). Of some physical interest is the closure $\mathbf{R}_M$ of the n-step Markov processes in the $\bar{d}$-metric. An n-step Markov process is the product of a Bernoulli shift and a finite rotation [2]. If two such processes are close enough in the $\bar{d}$-metric their rotation factors must coincide. Hence the class $\mathbf{R}_M$ is just the class of direct products of Bernoulli shifts with finite rotations and is therefore regular.

It would be of much interest to know whether some further classes of interest are regular. Among these are the following:

(a) The class of products of rotations of the circle and Bernoulli shifts. (This clearly satisfies (R1) (R2) and (R4)).

(b) The class of affine maps of a torus, or some other group of interest.

Simple physical descriptions of such classes (such as that of $\mathbf{R}_M$ as the $\bar{d}$-closure of the n-step Markov processes) as well as further properties of such classes need investigating (e.g., does (R2) imply (R4)?).

REFERENCES

[1] Adler, R. (1964). Invariant and reducing subalgebras of measure preserving transformations. *Trans. Amer. Math. Soc.* **110** 350–360.

[2] Adler, R., Shields, P. and Smorodinsky, M. (1972). Irreducible Markov shifts. *Ann. Math. Statist.* **43** 1027–1029.

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[1] Professor Ornstein's paper was presented as a Special Invited Paper on June 20, 1972, at the Western Regional meeting of the Institute of Mathematical Statistics in Seattle, Washington. Professors P. C. Shields and R. M. Blumenthal were invited discussants at this meeting. The wider discussion contained herein was made possible by distributing advance copies of Professor Ornstein's manuscript to a group of interested persons. The Editor greatly appreciates Professor Ornstein's willing assistance in this as well as the interesting responses by the discussants.