DISCUSSION ON PROFESSOR KINGMAN'S PAPER

PROFESSOR D. L. BURKHOLDER (*University of Illinois*). The key to the pointwise ergodic theorem for subadditive stochastic processes is the decomposition

$$x_{st} = y_{st} + z_{st} \,. \tag{1}$$

Here $\mathbf{y}$ is an additive process satisfying $Ey_{01} = \gamma(\mathbf{x})$ and $\mathbf{z}$ is a nonnegative subadditive process with $\gamma(\mathbf{z}) = 0$. Kingman's elegant proof of the existence of such a decomposition for any subadditive process $\mathbf{x}$ is the most difficult part of his paper [2] so a slightly different proof, one which is more probabilistic in its orientation, may be of some interest.

The main novelty in the following proof is the use of Komlós's theorem [3]: *If $X_1, X_2, \cdots$ is an L^1-bounded random variable sequence* ($\sup_n E|X_n| < \infty$), *then there is a sequence $n_1 < n_2 < \cdots$ of positive integers and an integrable random variable Y such that*

$$j^{-1} \sum_{i=1}^{j} X_{n_i} \to Y$$

almost everywhere as $j \to \infty$. This theorem could be avoided if the sequence $\mathbf{f}_0 = (f_{0n})$ defined below could be shown to converge almost everywhere. However, quite apart from this possibility, Komlós's theorem gives at once enough information to carry through the proof of (1); it is enough to know that the sequence of Cesàro means of some subsequence of $\mathbf{f}_0$ converges almost everywhere.

The first steps leading to Komlós's remarkable theorem were made by Steinhaus, Austin, Rényi, and Révész; see [3]. Recent contributions have been made by Chatterji; for example, see [1].

Now let $\mathbf{x} = (x_{st})$ be a subadditive process and $\gamma = \gamma(\mathbf{x})$. The desired decomposition (1) may be deduced easily from the following fact.

LEMMA. *There is a stationary random variable sequence $f_0, f_1, \cdots$ such that $Ef_0 = \gamma$ and*

$$\sum_{k=s}^{t-1} f_k \leqq x_{st} \,, \qquad 0 \leqq s < t \,. \tag{2}$$

Given this, let $y_{st} = \sum_{k=s}^{t-1} f_k$ and $z_{st} = x_{st} - y_{st}$. Then $\mathbf{y}$ is an additive process with $Ey_{01} = \gamma$ and $\mathbf{z}$ is a nonnegative subadditive process with $\gamma(\mathbf{z}) = 0$:

$$\begin{aligned} t^{-1}Ez_{0t} &= t^{-1}E(x_{0t} - y_{0t}) \\ &= t^{-1}g_t - \gamma \to 0 \end{aligned}$$

as $t \to \infty$. This proves (1).

PROOF OF LEMMA. Let

$$f_{kn} = n^{-1} \sum_{r=1}^{n} (x_{k,k+r} - x_{k+1,k+r}) \,. \tag{3}$$

Since $(x_{s+1,t+1})$ has the same distribution as (x_{st}), it is clear that $\mathbf{f}_0 = (f_{0n})$, $\mathbf{f}_1 = (f_{1n}), \cdots$ is a stationary sequence.