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# BOOK REVIEW

H. DYM AND H. P. MCKEAN, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*. Prob. and Math. Statist. **31** Academic Press, New York, San Francisco, London, 1976, xi+333 pp., $35.00.

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Let $X$, $Y$ be two normal random variables with mean zero, variance 1, and covariance $\rho$. Given the information $X = c$, how can it be used to improve our knowledge of $Y$? This problem is easily solved. $Z = Y - \rho X$ is a normal random variable with mean zero and variance $1 - \rho^2$ and is independent of $X$. Therefore

$$P[a < Y < b \mid X = c] = P[a < Z + \rho c < b]$$
$$= (2\pi(1 - \rho^2))^{-\frac{1}{2}} \int_a^b \exp\left\{-\frac{(u - \rho c)^2}{2(1 - \rho^2)}\right\} du .$$

Note that whatever the value $c$, $EZ^2 = 1 - \rho^2 \leqq EY^2$ so that given $X$ the distribution of $Y$ becomes more concentrated depending upon the degree of correlation $\rho$. We note also that $\rho X = E(Y \mid X)$.

Similarly let $X_1, \cdots, X_n$ be normal random variables with mean zero and covariance $\Gamma_{ij}$, $i, j = 1, \cdots, n$. The distribution of $X_n$ given $X_1, \cdots, X_{n-1}$ is normal with mean $m = E(X_n \mid X_1, \cdots, X_{n-1})$ and variance $E((X_n - m)^2 \mid X_1, \cdots, X_{n-1}) = E(X_n - m)^2$. To see this consider the complex Hilbert space generated by sums $\eta = c_1 X_1 + \cdots + c_n X_n$ with norm $||\eta|| = (E|\eta|^2)^{\frac{1}{2}}$. Because of the equivalence between orthogonality of zero mean Gaussian random variables with respect to this norm and their statistical independence, $m$ is the orthogonal projection of $X_n$ onto the subspace spanned by $X_1, \cdots, X_{n-1}$; and $X_n - m$, being orthogonal to this subspace, is independent of the Borel field generated by $X_1, \cdots, X_{n-1}$. Utilizing the observations $X_1, \cdots, X_{n-1}$ to obtain the probability distribution of $X_n$ is what is meant by predicting $X_n$ given $X_1, \cdots, X_{n-1}$.

When the number of observations involved is finite (and the covariance function is known) the prediction problem is easily solved, but for an infinite number of observations it is more difficult. The earliest work was done independently by Kolmogorov in 1939 and 1941 (references not included here can be found in Dym and McKean's book) and by Wiener in 1942. Wiener's 1942 report, the so-called "yellow peril," was classified as a military secret and was finally published openly as *Extrapolation, Interpolation and Smoothing of Stationary Time Series* in 1949. Wiener was unaware of Kolmogorov's work. The problem that motivated Wiener [3] and perhaps also Kolmogorov was that of automatic fire control for anti-aircraft batteries. Roughly speaking, one observes some