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A CONTROL PROBLEM ARISING IN THE SEQUENTIAL DESIGN OF EXPERIMENTS

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1. Introduction and summary

The Pelé problem. Starting from an initial point $\mathbf{x}$ not in his playing field, a football player is to dribble onto the field. Due to irregularities in the surface on which the player is dribbling, and perhaps also to small inconsistencies in his kick, the movement of the ball has a "random" component; moreover, a kick with the left foot tends to have a somewhat different effect than a kick with the right foot. The player's objective is to move the ball onto the playing field with as few kicks as possible.

To make the problem more precise, suppose that the "playing field" is the first quadrant $\mathscr{Q}_+$ of $\mathbb{R}^2$. Assume that a kick with the left foot results in a (random) displacement whose distribution is F_A, while a kick with the right foot results in a (random) displacement distributed according to a law F_B (different from F_A). Assume also that F_A and F_B have finite second moments, and mean vectors $\boldsymbol{\mu}_A$ and $\boldsymbol{\mu}_B$ satisfying

$$\mu_A^{(1)} > \mu_B^{(1)} \geq 0, \qquad \mu_B^{(2)} > \mu_A^{(2)} \geq 0. \tag{1.1}$$

Then the Pelé problem may be phrased as an optimal control problem: Among all nonanticipating policies (a "policy" being a rule for deciding, at each step, which foot to kick with), find the policy $\mathscr{P}$, which minimizes $E_{\mathbf{x}}^{\mathscr{P}}T$, where T is the (random) number of kicks made before the first entry into $\mathscr{Q}_+$.

The problem as stated is, of course, a problem of "Markovian decision theory" [cf., for example, Derman (1970)], and may be "solved" by dynamic programming. But given that football players generally have neither the time nor adequate computational facilities for backward inductions, it seems reasonable to ask whether there is some simple policy which, although not optimal, is "nearly" optimal in some sense. One class of rules, which is particularly appealing, is the class of "straight-line switching" (SLS) policies: An SLS policy calls for a kick with the left foot whenever the ball is *above* a certain line (or ray) and a kick with the right whenever the ball is below this line.

We will prove

THEOREM A. *Suppose F_A and F_B have finite second moments and mean vectors satisfying* (1.1). *If $\mathscr{P}$ is any straight-line switching policy whose switch-*

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