The Annals of Probability
1993, Vol. 21, No. 1, 581–587

BOOK REVIEW

H. KUNITA, *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge, 1990, 346 pages, $69.50.

REVIEW BY T. E. HARRIS

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A stochastic flow is a family of random mappings $\phi_{s,t}$, $0 \le s \le t \le T$, of R^d (or other space) into itself, satisfying the composition relation $\phi_{t,u} \circ \phi_{s,t} = \phi_{s,u}$ if $s \le t \le u$. There are interesting cases with coalescence, but most treatments have dealt with homeomorphic or diffeomorphic flows. Then $\bar{\phi}$ defined by $\bar{\phi}_{s,t} = \phi_{s,t}$ for $s \le t$ and $\bar{\phi}_{s,t} = (\phi_{t,s})^{-1}$ for $s \ge t$, is a flow satisfying the composition relation for $0 \le s, t \le T$. If ϕ_t, $t \ge 0$ is a homeomorphism-valued process, then $\phi_{s,t} := \phi_t \circ (\phi_s)^{-1}$, $0 \le s, t \le T$ is a flow. If ϕ is a flow defined for $0 \le s, t \le T$, the restriction to $s \le t$ is called a *forward* flow, and is usually considered for fixed s. The restriction to $s \ge t$ is a *backward* flow. We speak mostly of forward flows, but there are parallel backward results, and the interplay between the two is important. ϕ_t denotes $\phi_{0,t}$.

In an important case $\phi_{s,t}(x)$ is continuous in s, t and x, and $s_1 \le t_1 \le s_2 \le t_2 \le \cdots$ implies that $\phi_{s_1,t_1}, \phi_{s_2,t_2}, \dots$ are independent mappings; such flows are called *Brownian*. For Brownian flows with some regularity conditions, the paths of a set of k points are a dk-dimensional diffusion. In the time-homogeneous case, the law of $\phi_{s,t}$ depends on $t - s$.

It is known that Brownian stochastic flows arise as solutions of Itô systems in R^d:

$$
(1)\qquad
\begin{aligned}
d\phi_{s,t}^i(x) &= \sum_{j=1}^{m} \sigma_j^i(\phi_{s,t}(x), t)\, dW_{jt} + b^i(\phi_{s,t}(x), t)\, dt, \qquad t \ge s;\\
\phi_{s,s}^i(x) &= x^i, \qquad 1 \le i \le d.
\end{aligned}
$$

Here the W's are independent Wiener processes in R^1. The σ's and b's satisfy familiar conditions. If we change σ without changing $\sigma\sigma^T$, the one-point motions are unchanged but in general the flow will be different. However, the flow is determined by the two-point motions, or alternatively by the *infinitesimal mean* (drift) $b(x, t)$ and the *infinitesimal covariance matrix*:

$$
(2)\qquad
\begin{aligned}
a^{ij}(x, y, t) &= \sum_k \sigma_k^i(x, t)\sigma_k^j(y, t)\\
&= \lim_{u \downarrow t} E\{(\phi_{tu}^i(x) - x^i)(\phi_{tu}^j(y) - y^j)\}/(u - t),\\
b^i(x, t) &= \lim_{u \downarrow t} E\{\phi_{tu}^i(x) - x^i\}/(u - t).
\end{aligned}
$$

Received July 1992.