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BOOK REVIEW

Ross G. Pinsky, *Positive Harmonic Functions and Diffusion: An Integrated Analytic and Probabilistic Approach.*

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The interplay between probability theory and partial differential equations has inspired some beautiful mathematics, including the martingale problems of Stroock and Varadhan, the Harnack inequality of Krylov and Safonov, the Malliavin calculus, and the application of Dirichlet forms to symmetric Markov processes. One of the principal ways the relationship comes about is as follows. Let $d \geq 2$ be the dimension, σ: $\mathbb{R}^d \to \mathbb{R}^{d\times d}$ a matrix-valued function, b: $\mathbb{R}^d \to \mathbb{R}^d$ a vector-valued function, and W_t a d-dimensional Brownian motion. Let X_t be the solution to the stochastic differential equation

$$dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt, \tag{1}$$

which means

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t b(X_s)\,ds,$$

where the first integral is the Itô stochastic integral. Let $a = \sigma\sigma^T$, where σ^T denotes the transpose of σ, and let $\mathscr{L}$ be the operator defined on C^2 functions by

$$\mathscr{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x). \tag{2}$$

We illustrate the connections between X_t and $\mathscr{L}$ by three examples. Suppose that σ and b are smooth functions of x.

(1) Suppose D is an open domain in $\mathbb{R}^d$ with a smooth boundary. Let f be a continuous function on the boundary of D. A function h is harmonic in D if h is C^2 in D and $\mathscr{L}h = 0$ in D. The Dirichlet problem is to find a harmonic function h on D that is continuous on the closure of D and agrees with f on the boundary of D. If τ_D is the first time the process X_t exits D, the solution to the Dirichlet problem can be given simply by

$$h(x) = \mathbb{E}^x f(X_{\tau_D}),$$

where $\mathbb{E}^x$ denotes the expectation with respect to the law of the solution to (1) when $X_0 = x$.

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