The limits of x under Σ being infinite, $x+1$ can be replaced by x, consequently

$$w_z(y) = \frac{n-z}{n} w_z(y-1).$$

This difference-equation, in which y is the variable, may easily be integrated. As we have, further,

$$w_z(0) = (-1)^z \beta_a(z),$$

we get

$$w_z(y) = (-1)^z \cdot \beta_a(z) \cdot \left(\frac{n-z}{n}\right)^y.$$

By Oppermann's inverse transformation we find now:

$$u_x(y) = (-1)^x \Sigma \beta_z(x) \cdot (-1)^z \cdot \beta_a(z) \cdot \left(\frac{n-z}{n}\right)^y,$$

Σ taken from $x = -\infty$ to $x = +\infty$. This expression

$$u_x(y) = \beta_a(x) \Sigma (-1)^{z+x} \cdot \beta_{a-x}(z-x) \cdot \left(\frac{n-z}{n}\right)^y$$

has the above mentioned practical short-comings, which are sensible particularly if n, $a-x$, or y are large numbers; in these cases an artifice like that used by Laplace (problem 17) becomes necessary. But our exact solution has a simple interpretation. The sum that multiplies $\beta_a(x)$ in $u_x(y)$, is the $(a-x)^{\text{th}}$ difference of the function $\left(\frac{n-z}{n}\right)^y$, and is found by a table of the values $\left(\frac{n-a}{n}\right)^y$, $\left(\frac{n-a+1}{n}\right)^y$, ... $\left(\frac{n-x-1}{n}\right)^y$, $\left(\frac{n-x}{n}\right)^y$, as the final difference formed by all these consecutive values. We learn from this interpretation that it is possible, if not easy, to solve this problem without the integration of any difference-equation, in a way analogous to that used in § 67, example 4.

If we make use of $w_z(y)$ to give us the half-invariants μ_1, μ_2 for the same law of errors as is expressed by $u_x(y)$, then we find for the mean value of x after y drawings

$$\lambda_1(y) = a\left(\frac{n-1}{n}\right)^y$$

and for the square of the mean error

$$\lambda_2(y) = a\left(\left(\frac{n-1}{n}\right)^y - \left(\frac{n-2}{n}\right)^y\right) + a^2\left(\left(\frac{n-2}{n}\right)^y - \left(\frac{n-1}{n}\right)^{2y}\right).$$

XVI. THE DETERMINATION OF PROBABILITIES A PRIORI AND A POSTERIORI.

§ 70. The computations of probabilities with which we have been dealing in the foregoing chapters have this point in common that we always assume one or several probabilities to be given, and then deduce from them the required ones. If now we ask, how